The shape of spherical rational quartics

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Rational curves on quadric surfaces can be seen as solutions to certain Diophantine equations in the ring of polynomials. In the case of the sphere $\mathbb{S}^2$, which is a representative of the class of oval quadrics, this equation takes the form $w^2 = x^2 + y^2 + z^2$. All irreducible solutions can be generated with the help of a classical representation formula from number theory, which was first noted by V.A. Lebesgue in 1868 [1]. More recently, this formula has been used to define a mapping from real projective 3–space onto the unit sphere, $\delta : P^3(\mathbb{R}) \to \mathbb{S}^2$, which has been called the *generalized stereographic projection* [2]. Due to its algebraic origin, this mapping can be used to generate any rational curve of degree $2n$ on the sphere as the image of a curve of degree n.

This mapping can be discussed from a geometrical point of view, too. It can be shown to identify the points of the unit sphere with a special two–parameter system of lines, called a elliptic linear congruence. (See [3] for more information on line geometry). As a major advantage, this mapping avoids the dependency on the choice of the center of projection, which is always present for the standard stereographic projection.

Spherical curves have applications in geometric modelling and in kinematics and animation. Curves on the 4D unit sphere can be identified with spherical (i.e., rotational) motions, and they can therefore be used for computer animation and motion planning.

We use the generalized stereographic projection to generate and to analyze the solutions to the C^1 Hermite interpolation problem with spherical rational curves on the sphere $\mathbb{S}^2$. Given two points with associated first derivatives on a sphere, we interpolate these data with a rational curve segment of degree 4. (In the 4D case, the data span a three–dimensional space, and is natural to ask for solutions which are contained in it. Note that also non–3D solutions exist; they have no singularities.)

The data can be interpolated with a two–parameter family of solutions. Using the generalized stereographic projection, each solution can be identified with a point in a certain parameter plane. We discuss the shape of the solutions, which is characterized by the presence of cusps or double points. This results in a so–called *characterization diagram:* the parameter plane is subdivided in different regions which correspond to solutions exhibiting the same shape.

This talk is based on joint work with Wenping Wang (The University of Hong Kong, China).

[1] L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, New York, 1952.

[2] R. Dietz, J. Hoschek, and B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, Computer Aided Geometric Design 10 (1993), 211–229.

[3] H. Pottmann and J. Wallner, Computational Line Geometry, Springer, 2001.