We have to prove:

$$\underset{X}{\forall}\, \mathtt{IsSorted}[X] \Rightarrow \left(\underset{a}{\forall}\, (\mathcal{M}[\mathtt{Insert}[a, X]] == (\{\{a\}\} \uplus \mathcal{M}[X])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a, X]]\right) \qquad \text{(insert–sorted)}$$

under the assumptions:

$$\mathtt{IsSorted}[\epsilon], \qquad \text{(sorted–empty–tree)}$$

$$\underset{X}{\forall}\, X \leq \epsilon, \qquad \text{(smaller–than–empty–tree)}$$

$$\underset{X}{\forall}\, \epsilon \leq X, \qquad \text{(empty–tree–smaller–than)}$$

$$\underset{L}{\forall}\underset{a}{\forall}\underset{R}{\forall}\, \mathtt{IsSorted}[\langle L, a, R\rangle] \Leftrightarrow (\mathtt{IsSorted}[L] \wedge \mathtt{IsSorted}[R] \wedge (L \leq a) \wedge (a \leq R)), \qquad \text{(sorted–tree)}$$

$$\underset{L}{\forall}\underset{a}{\forall}\underset{R}{\forall}\underset{b}{\forall}\, (\langle L, a, R\rangle \leq b) \Leftrightarrow ((L \leq b) \wedge (a \leq b) \wedge (R \leq b)), \qquad \text{(tree–less–elem)}$$

$$\underset{L}{\forall}\underset{a}{\forall}\underset{R}{\forall}\underset{b}{\forall}\, (b \leq \langle L, a, R\rangle) \Leftrightarrow ((b \leq L) \wedge (b \leq a) \wedge (b \leq R)), \qquad \text{(elem–less–tree)}$$

$$\underset{a}{\forall}\underset{X}{\forall}\underset{b}{\forall}\, (\mathtt{Insert}[a, X] \leq b) \Leftrightarrow ((a \leq b) \wedge (X \leq b)), \qquad \text{(insert–less–than)}$$

$$\underset{a}{\forall}\underset{X}{\forall}\underset{b}{\forall}\, (b \leq \mathtt{Insert}[a, X]) \Leftrightarrow ((b \leq a) \wedge (b \leq X)), \qquad \text{(elem–less–insert)}$$

$$\underset{L}{\forall}\underset{a}{\forall}\underset{R}{\forall}\, \mathcal{M}[\langle L, a, R\rangle] == (\mathcal{M}[L] \uplus \{\{a\}\} \uplus \mathcal{M}[R]), \qquad \text{(full–tree–multiset–direct)}$$

$$\underset{L}{\forall}\underset{a}{\forall}\underset{R}{\forall}\, \mathcal{M}[\langle L, a, R\rangle] == (\mathcal{M}[R] \uplus \{\{a\}\} \uplus \mathcal{M}[L]), \qquad \text{(full–tree–multiset–reverse)}$$

$$\underset{a}{\forall}\, \mathcal{M}[\langle \epsilon, a, \epsilon\rangle] == \{\{a\}\}, \qquad \text{(unit–tree–multiset)}$$

$$\mathcal{M}[\epsilon] == \emptyset, \qquad \text{(empty–tree–multiset)}$$

$$\underset{A}{\forall}\, (A \uplus \emptyset) == A, \qquad \text{(empty–multiset–in–union)}$$

$$\underset{A}{\forall}\underset{B}{\forall}\underset{C}{\forall}\, ((A \uplus B) \uplus C) == (A \uplus B \uplus C), \qquad \text{(union–associativiy)}$$

$$\underset{A}{\forall}\underset{B}{\forall}\, (A \uplus B) == (B \uplus A). \qquad \text{(union–commutativity)}$$

Proof by algorithm constructor.

For proving the universal goal (insert–sorted), take $X0$ a.b.f. and prove:

$$\mathtt{IsSorted}[X0] \Rightarrow \left(\underset{a}{\forall}\, (\mathcal{M}[\mathtt{Insert}[a, X0]] == (\{\{a\}\} \uplus \mathcal{M}[X0])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a, X0]]\right). \qquad \text{(G\#163)}$$

For proving (G#163), use alternatively the cover set $\{\epsilon, \langle L, a, R\rangle\}$ or no cover set for the Skolem constant $X0$.

⚠ Alternative 1: cover set $\{\epsilon, \langle L, a, R\rangle\}$.

Cover set cases:

⚠ Case 1: $X0 = \epsilon$. The goal becomes:

$$\mathtt{IsSorted}[\epsilon] \Rightarrow \left(\underset{a}{\forall}\, (\mathcal{M}[\mathtt{Insert}[a, \epsilon]] == (\{\{a\}\} \uplus \mathcal{M}[\epsilon])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a, \epsilon]]\right). \qquad \text{(G\#163.1)}$$

Implicative goal (G#163.1) is split. Assume:

$$\mathtt{IsSorted}[\epsilon], \qquad \text{(A\#182)}$$

and prove:

$$\underset{a}{\forall}\, (\mathcal{M}[\mathtt{Insert}[a, \epsilon]] == (\{\{a\}\} \uplus \mathcal{M}[\epsilon])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a, \epsilon]], \qquad \text{(G\#183)}$$

For proving the universal goal (G#183), take *a0* a.b.f. and prove:

$$(\mathcal{M}[\texttt{Insert}[a0, \epsilon]] == (\{\{a0\}\} \uplus \mathcal{M}[\epsilon])) \wedge \texttt{IsSorted}[\texttt{Insert}[a0, \epsilon]]. \quad \text{(G\#201)}$$

Using "empty-tree-multiset", the goal (G#201) is simplified to:

$$(\mathcal{M}[\texttt{Insert}[a0, \epsilon]] == (\{\{a0\}\} \uplus \emptyset)) \wedge \texttt{IsSorted}[\texttt{Insert}[a0, \epsilon]]. \quad \text{(G\#202)}$$

Using "empty-multiset-in-union", the goal (G#202) is simplified to:

$$(\mathcal{M}[\texttt{Insert}[a0, \epsilon]] == \{\{a0\}\}) \wedge \texttt{IsSorted}[\texttt{Insert}[a0, \epsilon]]. \quad \text{(G\#203)}$$

Using goal (G#203), the solution is: Insert[$a0$, ϵ] == $\langle \epsilon, a0, \epsilon \rangle$, and the goal is reduced to:

$$\texttt{IsSorted}[\langle \epsilon, a0, \epsilon \rangle]. \quad \text{(G\#204)}$$

Using "sorted-tree", the goal (G#204) is simplified to:

$$\texttt{IsSorted}[\epsilon] \wedge (\epsilon \le a0) \wedge (a0 \le \epsilon). \quad \text{(G\#205)}$$

Using "A#182", the goal (G#205) is simplified to:

$$(\epsilon \le a0) \wedge (a0 \le \epsilon). \quad \text{(G\#206)}$$

Using "empty-tree-smaller-than", the goal (G#206) is simplified to:

$$a0 \le \epsilon. \quad \text{(G\#207)}$$

Using "smaller-than-empty-tree", the goal (G#207) is true: success:

⚠ Case 2: $X0 = \langle L0, a0, R0 \rangle$ with a.b.f. $L0$, $a0$, and $R0$. The goal becomes:

$$\texttt{IsSorted}[\langle L0, a0, R0 \rangle] \Rightarrow \left(\forall_{a} (\mathcal{M}[\texttt{Insert}[a, \langle L0, a0, R0 \rangle]] == (\{\{a\}\} \uplus \mathcal{M}[\langle L0, a0, R0 \rangle])) \wedge \texttt{IsSorted}[\texttt{Insert}[a, \langle L0, a0, R0 \rangle]]\right). \quad \text{(G\#163.2)}$$

Implicative goal (G#163.2) is split. Assume:

$$\texttt{IsSorted}[\langle L0, a0, R0 \rangle], \quad \text{(A\#209)}$$

and prove:

$$\forall_{a} (\mathcal{M}[\texttt{Insert}[a, \langle L0, a0, R0 \rangle]] == (\{\{a\}\} \uplus \mathcal{M}[\langle L0, a0, R0 \rangle])) \wedge \texttt{IsSorted}[\texttt{Insert}[a, \langle L0, a0, R0 \rangle]], \quad \text{(G\#210)}$$

The assumption (A#209) is simplified using "sorted-tree" to:

$$\texttt{IsSorted}[L0] \wedge \texttt{IsSorted}[R0] \wedge (L0 \le a0) \wedge (a0 \le R0) \wedge (L0 \le R0). \quad \text{(A\#227)}$$

For proving the universal goal (G#210), take $\overline{a1}$ a.b.f. and prove:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0 \rangle]] == (\{\{\overline{a1}\}\} \uplus \mathcal{M}[\langle L0, a0, R0 \rangle])) \wedge \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0 \rangle]]. \quad \text{(G\#245)}$$

Using "full-tree-multiset-direct", the goal (G#245) is expanded to:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0 \rangle]] == (\{\{\overline{a1}\}\} \uplus (\{\{a0\}\} \uplus \mathcal{M}[L0] \uplus \mathcal{M}[R0]))) \wedge \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0 \rangle]]. \quad \text{(G\#341)}$$

Using "union-associativity-left", the goal (G#341) is simplified to:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]] == (\{\{\overline{a1}\}\} \uplus \{\{a0\}\} \uplus \mathcal{M}[L0] \uplus \mathcal{M}[R0])) \land \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]]. \quad \text{(G\#342)}$$

The conjunction (A#227) is split into:

$$\texttt{IsSorted}[L0], \quad \text{(A\#227.1)}$$

$$\texttt{IsSorted}[R0], \quad \text{(A\#227.2)}$$

$$L0 \le a0, \quad \text{(A\#227.3)}$$

$$a0 \le R0, \quad \text{(A\#227.4)}$$

$$L0 \le R0. \quad \text{(A\#227.5)}$$

For proving (G#342), use alternatively the pairs of multisets ⟨1, 3⟩, ⟨2, 3⟩, ⟨1, 4⟩, and ⟨2, 4⟩.

⚠ Alternative 1, pair ⟨1, 3⟩: $\{\{\overline{a1}\}\} \uplus \mathcal{M}[L0]$.

By "union-associativity" and "union-commutativity", the goal becomes:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]] == ((\{\{\overline{a1}\}\} \uplus \mathcal{M}[L0]) \uplus \{\{a0\}\} \uplus \mathcal{M}[R0])) \land \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]]. \quad \text{(G\#442)}$$

By Noetherian induction, using $L0 \prec X0$, assume:

$$\texttt{IsSorted}[L0] \Rightarrow \left(\underset{a}{\forall} (\mathcal{M}[\texttt{Insert}[a, L0]] == (\{\{a\}\} \uplus \mathcal{M}[L0])) \land \texttt{IsSorted}[\texttt{Insert}[a, L0]]\right), \quad \text{(A\#446)}$$

from which by (A#227.1) follows:

$$\underset{a}{\forall} (\mathcal{M}[\texttt{Insert}[a, L0]] == (\{\{a\}\} \uplus \mathcal{M}[L0])) \land \texttt{IsSorted}[\texttt{Insert}[a, L0]], \quad \text{(A\#447)}$$

which is instantiated to:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, L0]] == (\{\{\overline{a1}\}\} \uplus \mathcal{M}[L0])) \land \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, L0]], \quad \text{(A\#448)}$$

and the goal becomes:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]] == (\mathcal{M}[\texttt{Insert}[\overline{a1}, L0]] \uplus \{\{a0\}\} \uplus \mathcal{M}[R0])) \land \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]]. \quad \text{(G\#449)}$$

The goal (G#449) is simplified using several alternatives.

⚠ Using full-tree-multiset-direct, the goal (G#449) is simplified to:

$$(\mathcal{M}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]] == \mathcal{M}[\langle \texttt{Insert}[\overline{a1}, L0], a0, R0\rangle]) \land \texttt{IsSorted}[\texttt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]]. \quad \text{(G\#450)}$$

Using goal (G#450), the solution is: $\text{Insert}[\overline{a1}, \langle L0, a0, R0\rangle] == \langle \text{Insert}[\overline{a1}, L0], a0, R0\rangle$, and the goal is reduced to:

$$\texttt{IsSorted}[\langle \texttt{Insert}[\overline{a1}, L0], a0, R0\rangle]. \quad \text{(G\#590)}$$

Using "sorted-tree", the goal (G#590) is simplified to:

$$\texttt{IsSorted}[\texttt{Insert}[\overline{a1}, L0]] \land \texttt{IsSorted}[R0] \land (\texttt{Insert}[\overline{a1}, L0] \le a0) \land (a0 \le R0). \quad \text{(G\#591)}$$

Using "A#227.4", the goal (G#591) is simplified to:

$$\mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, L0]] \wedge \mathtt{IsSorted}[R0] \wedge (\mathtt{Insert}[\overline{a1}, L0] \leq a0). \quad \text{(G\#592)}$$

Using "A#227.2", the goal (G#592) is simplified to:

$$\mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, L0]] \wedge (\mathtt{Insert}[\overline{a1}, L0] \leq a0). \quad \text{(G\#593)}$$

Using "insert-less-elem", the goal (G#593) is simplified to:

$$\mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, L0]] \wedge ((\overline{a1} \leq a0) \wedge (L0 \leq a0)). \quad \text{(G\#594)}$$

Using "A#227.3", the goal (G#594) is simplified to:

$$\mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, L0]] \wedge (\overline{a1} \leq a0). \quad \text{(G\#595)}$$

The conjunction (A#448) is split into:

$$\mathcal{M}[\mathtt{Insert}[\overline{a1}, L0]] == (\{\{\overline{a1}\}\} \uplus \mathcal{M}[L0]), \quad \text{(A\#448.1)}$$

$$\mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, L0]]. \quad \text{(A\#448.2)}$$

Using "A#448.2", the goal (G#595) is simplified to:

$$\overline{a1} \leq a0. \quad \text{(G\#645)}$$

The goal is used as a condition in the following clause of the algorithm:

$$(\overline{a1} \leq a0) \Rightarrow (\mathtt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle] == \langle \mathtt{Insert}[\overline{a1}, L0], a0, R0\rangle). \quad \text{(A\#646)}$$

Success.

⚠ Using full-tree-multiset-reverse, the goal (G#449) is simplified to:

$$(\mathcal{M}[\mathtt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]] == \mathcal{M}[\langle R0, a0, \mathtt{Insert}[\overline{a1}, L0]\rangle]) \wedge \mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]]. \quad \text{(G\#451)}$$

The solution Insert[$\bar{a}1$, ⟨$L0$, $a0$, $R0$⟩] == ⟨$R0$, $a0$, Insert[$\bar{a}1$, $L0$]⟩ is not admissible because $R0$ and $L0$ are in the wrong order according to (A#227.5)

⚠ Alternative 2, pair ⟨2, 3⟩: {{$a0$}} ⊎ $\mathcal{M}$[$L0$].

By "union-associativity" and "union-commutativity", the goal becomes:

$$(\mathcal{M}[\mathtt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]] == ((\{\{a0\}\} \uplus \mathcal{M}[L0]) \uplus \{\{\overline{a1}\}\} \uplus \mathcal{M}[R0])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[\overline{a1}, \langle L0, a0, R0\rangle]]. \quad \text{(G\#443)}$$

By Noetherian induction, using $L0 \prec X0$, assume:

$$\mathtt{IsSorted}[L0] \Rightarrow (\forall_a (\mathcal{M}[\mathtt{Insert}[a, L0]] == (\{\{a\}\} \uplus \mathcal{M}[L0])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a, L0]]), \quad \text{(A\#647)}$$

from which by (A#227.1) follows:

$$\forall_a (\mathcal{M}[\mathtt{Insert}[a, L0]] == (\{\{a\}\} \uplus \mathcal{M}[L0])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a, L0]], \quad \text{(A\#648)}$$

which is instantiated to:

$$(\mathcal{M}[\mathtt{Insert}[a0, L0]] == (\{\{a0\}\} \uplus \mathcal{M}[L0])) \wedge \mathtt{IsSorted}[\mathtt{Insert}[a0, L0]], \quad \text{(A\#649)}$$

and the goal becomes:

$$\left(\mathcal{M}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right] == \left(\mathcal{M}[\text{Insert}[a0, L0]] \uplus \left\{\{\bar{a1}\}\right\} \uplus \mathcal{M}[R0]\right)\right) \wedge \text{IsSorted}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right]. \quad \text{(G\#650)}$$

The goal (G#650) is simplified using several alternatives.

⚠ Using full-tree-multiset-direct, the goal (G#650) is simplified to:

$$\left(\mathcal{M}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right] == \mathcal{M}\left[\left\langle \text{Insert}[a0, L0], \bar{a1}, R0\right\rangle\right]\right) \wedge \text{IsSorted}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right]. \quad \text{(G\#651)}$$

Using goal (G#651), the solution is: $\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle] == \langle \text{Insert}[a0, L0], \bar{a1}, R0\rangle$, and the goal is reduced to:

$$\text{IsSorted}\left[\left\langle \text{Insert}[a0, L0], \bar{a1}, R0\right\rangle\right]. \quad \text{(G\#791)}$$

Using "sorted-tree", the goal (G#791) is simplified to:

$$\text{IsSorted}[\text{Insert}[a0, L0]] \wedge \text{IsSorted}[R0] \wedge \left(\text{Insert}[a0, L0] \le \bar{a1}\right) \wedge \left(\bar{a1} \le R0\right). \quad \text{(G\#792)}$$

Using "A#227.2", the goal (G#792) is simplified to:

$$\text{IsSorted}[\text{Insert}[a0, L0]] \wedge \left(\text{Insert}[a0, L0] \le \bar{a1}\right) \wedge \left(\bar{a1} \le R0\right). \quad \text{(G\#793)}$$

Using "insert-less-elem", the goal (G#793) is simplified to:

$$\text{IsSorted}[\text{Insert}[a0, L0]] \wedge \left(\left(a0 \le \bar{a1}\right) \wedge \left(L0 \le \bar{a1}\right)\right) \wedge \left(\bar{a1} \le R0\right). \quad \text{(G\#794)}$$

The conjunction (A#649) is split into:

$$\mathcal{M}[\text{Insert}[a0, L0]] == \left(\{\{a0\}\} \uplus \mathcal{M}[L0]\right), \quad \text{(A\#649.1)}$$

$$\text{IsSorted}[\text{Insert}[a0, L0]]. \quad \text{(A\#649.2)}$$

Using "A#649.2", the goal (G#794) is simplified to:

$$\left(\left(a0 \le \bar{a1}\right) \wedge \left(L0 \le \bar{a1}\right)\right) \wedge \left(\bar{a1} \le R0\right). \quad \text{(G\#844)}$$

Proof fails.

⚠ Using full-tree-multiset-reverse, the goal (G#650) is simplified to:

$$\left(\mathcal{M}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right] == \mathcal{M}\left[\left\langle R0, \bar{a1}, \text{Insert}[a0, L0]\right\rangle\right]\right) \wedge \text{IsSorted}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right]. \quad \text{(G\#652)}$$

The solution $\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle] == \langle R0, \bar{a1}, \text{Insert}[a0, L0]\rangle$ is not admissible because $R0$ and $L0$ are in the wrong order according to (A#227.5)

⚠ Alternative 3, pair ⟨1, 4⟩: $\{\{\bar{a1}\}\} \uplus \mathcal{M}[R0]$.

By "union-associativity" and "union-commutativity", the goal becomes:

$$\left(\mathcal{M}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right] == \left(\left(\left\{\{\bar{a1}\}\right\} \uplus \mathcal{M}[R0]\right) \uplus \{\{a0\}\} \uplus \mathcal{M}[L0]\right)\right) \wedge \text{IsSorted}\left[\text{Insert}\left[\bar{a1}, \langle L0, a0, R0\rangle\right]\right]. \quad \text{(G\#444)}$$

By Noetherian induction, using $R0 \prec X0$, assume:

$$\text{IsSorted}[R0] \Rightarrow \left(\underset{a}{\forall}\left(\mathcal{M}[\text{Insert}[a, R0]] == (\{\{a\}\} \uplus \mathcal{M}[R0])\right) \wedge \text{IsSorted}[\text{Insert}[a, R0]]\right), \tag{A\#845}$$

from which by (A#227.2) follows:

$$\underset{a}{\forall}\left(\mathcal{M}[\text{Insert}[a, R0]] == (\{\{a\}\} \uplus \mathcal{M}[R0])\right) \wedge \text{IsSorted}[\text{Insert}[a, R0]], \tag{A\#846}$$

which is instantiated to:

$$\left(\mathcal{M}[\text{Insert}[\bar{a1}, R0]] == \left(\{\{\bar{a1}\}\} \uplus \mathcal{M}[R0]\right)\right) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, R0]], \tag{A\#847}$$

and the goal becomes:

$$\left(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == \left(\mathcal{M}[\text{Insert}[\bar{a1}, R0]] \uplus \{\{a0\}\} \uplus \mathcal{M}[L0]\right)\right) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]]. \tag{G\#848}$$

The goal (G#848) is simplified using several alternatives.

⚠ Using full-tree-multiset-direct, the goal (G#848) is simplified to:

$$\left(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == \mathcal{M}[\langle \text{Insert}[\bar{a1}, R0], a0, L0\rangle]\right) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]]. \tag{G\#849}$$

The solution Insert[$\bar{a1}$, ⟨$L0$, $a0$, $R0$⟩] == ⟨Insert[$\bar{a1}$, $R0$], $a0$, $L0$⟩ is not admissible because $R0$ and $L0$ are in the wrong order according to (A#227.5)

⚠ Using full-tree-multiset-reverse, the goal (G#848) is simplified to:

$$\left(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == \mathcal{M}[\langle L0, a0, \text{Insert}[\bar{a1}, R0]\rangle]\right) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]]. \tag{G\#850}$$

Using goal (G#850), the solution is: Insert[$\bar{a1}$, ⟨$L0$, $a0$, $R0$⟩] == ⟨$L0$, $a0$, Insert[$\bar{a1}$, $R0$]⟩, and the goal is reduced to:

$$\text{IsSorted}[\langle L0, a0, \text{Insert}[\bar{a1}, R0]\rangle]. \tag{G\#989}$$

Using "sorted-tree", the goal (G#989) is simplified to:

$$\text{IsSorted}[L0] \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, R0]] \wedge (L0 \le a0) \wedge (a0 \le \text{Insert}[\bar{a1}, R0]). \tag{G\#990}$$

Using "A#227.3", the goal (G#990) is simplified to:

$$\text{IsSorted}[L0] \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, R0]] \wedge (a0 \le \text{Insert}[\bar{a1}, R0]). \tag{G\#991}$$

Using "A#227.1", the goal (G#991) is simplified to:

$$\text{IsSorted}[\text{Insert}[\bar{a1}, R0]] \wedge (a0 \le \text{Insert}[\bar{a1}, R0]). \tag{G\#992}$$

Using "elem-less-insert", the goal (G#992) is simplified to:

$$\text{IsSorted}[\text{Insert}[\bar{a1}, R0]] \wedge \left((a0 \le \bar{a1}) \wedge (a0 \le R0)\right). \tag{G\#993}$$

Using "A#227.4", the goal (G#993) is simplified to:

$$\text{IsSorted}[\text{Insert}[\bar{a1}, R0]] \wedge (a0 \le \bar{a1}). \tag{G\#994}$$

The conjunction (A#847) is split into:

$\mathcal{M}[\text{Insert}[\bar{a1}, R0]] == (\{\{\bar{a1}\}\} \uplus \mathcal{M}[R0]),$ (A#847.1)

$\text{IsSorted}[\text{Insert}[\bar{a1}, R0]].$ (A#847.2)

Using "A#847.2", the goal (G#994) is simplified to:

$a0 \leq \bar{a1}.$ (G#1044)

The goal is used as a condition in the following clause of the algorithm:

$(a0 \leq \bar{a1}) \Rightarrow (\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle] == \langle L0, a0, \text{Insert}[\bar{a1}, R0]\rangle).$ (A#1045)

Success.

⚠ Alternative 4, pair ⟨2, 4⟩: $\{\{a0\}\} \uplus \mathcal{M}[R0]$.

By "union-associativity" and "union-commutativity", the goal becomes:

$(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == ((\{\{a0\}\} \uplus \mathcal{M}[R0]) \uplus \{\{\bar{a1}\}\} \uplus \mathcal{M}[L0])) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]].$ (G#445)

By Noetherian induction, using $R0 \prec X0$, assume:

$\text{IsSorted}[R0] \Rightarrow (\forall_a (\mathcal{M}[\text{Insert}[a, R0]] == (\{\{a\}\} \uplus \mathcal{M}[R0])) \wedge \text{IsSorted}[\text{Insert}[a, R0]]),$ (A#1046)

from which by (A#227.2) follows:

$\forall_a (\mathcal{M}[\text{Insert}[a, R0]] == (\{\{a\}\} \uplus \mathcal{M}[R0])) \wedge \text{IsSorted}[\text{Insert}[a, R0]],$ (A#1047)

which is instantiated to:

$(\mathcal{M}[\text{Insert}[a0, R0]] == (\{\{a0\}\} \uplus \mathcal{M}[R0])) \wedge \text{IsSorted}[\text{Insert}[a0, R0]],$ (A#1048)

and the goal becomes:

$(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == (\mathcal{M}[\text{Insert}[a0, R0]] \uplus \{\{\bar{a1}\}\} \uplus \mathcal{M}[L0])) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]].$ (G#1049)

The goal (G#1049) is simplified using several alternatives.

⚠ Using full-tree-multiset-direct, the goal (G#1049) is simplified to:

$(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == \mathcal{M}[\langle \text{Insert}[a0, R0], \bar{a1}, L0\rangle]) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]].$ (G#1050)

The solution $\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle] == \langle \text{Insert}[a0, R0], \bar{a1}, L0\rangle$ is not admissible because $R0$ and $L0$ are in the wrong order according to (A#227.5)

⚠ Using full-tree-multiset-reverse, the goal (G#1049) is simplified to:

$(\mathcal{M}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]] == \mathcal{M}[\langle L0, \bar{a1}, \text{Insert}[a0, R0]\rangle]) \wedge \text{IsSorted}[\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle]].$ (G#1051)

Using goal (G#1051), the solution is: $\text{Insert}[\bar{a1}, \langle L0, a0, R0\rangle] == \langle L0, \bar{a1}, \text{Insert}[a0, R0]\rangle$, and the goal is reduced to:

$$\text{IsSorted}\left[\left\langle L0,\ \overline{a1},\ \text{Insert}[a0,\ R0]\right\rangle\right]. \tag{G\#1190}$$

Using "sorted-tree", the goal (G#1190) is simplified to:

$$\text{IsSorted}[L0] \wedge \text{IsSorted}[\text{Insert}[a0,\ R0]] \wedge \left(L0 \le \overline{a1}\right) \wedge \left(\overline{a1} \le \text{Insert}[a0,\ R0]\right). \tag{G\#1191}$$

Using "A#227.1", the goal (G#1191) is simplified to:

$$\text{IsSorted}[\text{Insert}[a0,\ R0]] \wedge \left(L0 \le \overline{a1}\right) \wedge \left(\overline{a1} \le \text{Insert}[a0,\ R0]\right). \tag{G\#1192}$$

Using "elem-less-insert", the goal (G#1192) is simplified to:

$$\text{IsSorted}[\text{Insert}[a0,\ R0]] \wedge \left(L0 \le \overline{a1}\right) \wedge \left(\left(\overline{a1} \le a0\right) \wedge \left(\overline{a1} \le R0\right)\right). \tag{G\#1193}$$

The conjunction (A#1048) is split into:

$$\mathcal{M}[\text{Insert}[a0,\ R0]] == (\{\{a0\}\} \uplus \mathcal{M}[R0]), \tag{A\#1048.1}$$

$$\text{IsSorted}[\text{Insert}[a0,\ R0]]. \tag{A\#1048.2}$$

Using "A#1048.2", the goal (G#1193) is simplified to:

$$\left(L0 \le \overline{a1}\right) \wedge \left(\left(\overline{a1} \le a0\right) \wedge \left(\overline{a1} \le R0\right)\right). \tag{G\#1243}$$

Proof fails (no applicable rule).

Goal:

⚠ Alternative 2: no cover set.

Implicative goal (G#163) is split. Assume:

$$\text{IsSorted}[X0], \tag{A\#1244}$$

and prove:

$$\underset{a}{\forall} \left(\mathcal{M}[\text{Insert}[a,\ X0]] == (\{\{a\}\} \uplus \mathcal{M}[X0])\right) \wedge \text{IsSorted}[\text{Insert}[a,\ X0]], \tag{G\#1245}$$

For proving the universal goal (G#1245), take *a0* a.b.f. and prove:

$$\left(\mathcal{M}[\text{Insert}[a0,\ X0]] == (\{\{a0\}\} \uplus \mathcal{M}[X0])\right) \wedge \text{IsSorted}[\text{Insert}[a0,\ X0]]. \tag{G\#1262}$$

By "union-associativity" and "union-commutativity", the goal becomes:

$$\left(\mathcal{M}[\text{Insert}[a0,\ X0]] == (\{\{a0\}\} \uplus \mathcal{M}[X0])\right) \wedge \text{IsSorted}[\text{Insert}[a0,\ X0]]. \tag{G\#1263}$$

Proof fails.