A Contribution to the Symmetry Classification Problem for 2nd Order PDEs in one Dependent and two Independent Variables

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Chapter 1

Introduction

This work deals with the most important means for finding closed form solutions of non-linear differential equations (DEs for short), the so called non-trivial symmetries that a given DE may admit. Apart from being used in finding exact solutions of DEs, symmetries also reveal important information about the properties of DEs; they can be used to verify and develop numerical schemes, to obtain conservation laws of a given DE, and even gauge theory is based on the continuous symmetries of certain relativistic fields.

The study of symmetries has been initiated by Sophus Lie. Roughly speaking, a symmetry is a point- or contact-transformation that does not change the form of a DE. Obviously, the entirety of symmetry transformations of any given DE forms a group. The term *symmetry group* is applied to the *largest* group of transformations with this property. The symmetry problem for DEs comes in various versions. On the one hand, there is the classification problem. It aims at obtaining a complete survey of all possible symmetries for a class of given DEs, e.g. DEs of a given order in a predetermined number of dependent and independent variables. We will be concerned with this version of the problem. The starting point of this approach is always a listing of groups whose differential invariants determine the general form of a DE that may be invariant under the respective group. On the other hand, the symmetry group of any given particular DE needs to be determined *algorithmically,* and may then be applied for simplifying the equation, finding special classes of solutions, etc. The key for solving the latter problem is the *Janet base* for the so-called determining system of the respective equation, since its coefficients determine the symmetry group. These two versions of the problem are closely related to each other.

Sophus Lie (1842-1899) determined all continuous transformation groups of the two dimensional plane [7], and gave normal forms for any ordinary DE that is invariant under one of those groups. Thereby, higher order equations are given implicitly, since they can be derived recursively from the corresponding equations of lower order.

We deal with the extension of this program to a special class of partial differential equations.

Problem 1 *We aim at a contribution to the symmetry classification problem for partial differential equations of order two in one dependent and two independent variables, i.e. to provide a listing of all possible symmetry groups that may be admitted by this family of equations.*

A complete strategy to tackle the whole classification problem is as follows:

1) List all finite continuous transformation groups of the three dimensional space in coordinates x, y, and z.

2) Find all differential invariants of the groups given in 1) where z depends on x and y. For a given group, the DEs invariant under this group are functions of these invariants.

3) Determine the group types, i.e. by using each invariant from 2), find criterions that allow to identify the symmetry group for a given DE from our classification.

In Chapter 2 ("Basic Theoretical Concepts"), we introduce the theory necessary to calculate differential invariants and basic theory of Lie algebras of vector fields. In Chapter 3 ("The Space Point Groups"), we give a listing of all continuous groups of the three dimensional space that is based on work by S. Lie [7, 8] and U. Amaldi [1], thereby solving point 1) of the above strategy. In Chapter 4 ("Differential Invariants of Order 2"), the differential invariants of most of the space groups listed in Chapter 3 are determined, thereby solving the second part of 2) to a large extent. In Chapter 5 ("Lower Invariants") the lower invariants for the same group classes as in Chapter 4 are provided. Finally, in Chapter 6 ("Examples and Conclusion") we consider some classical PDEs, indicate a way to compute higher invariants, and outline remaining work and open problems.

Due to the huge number of space groups, point 3) of the above outline is not considered in this work. This last step would be accomplished by computing the Janet base of the determining system of any differential invariant from 2), applying a general point transformation to it, which in general destroys the Janet base property, and reestablishing the Janet base property by applying the algorithm Janet base again. Thereby, a classification of Janet bases for determining systems of DEs for this class would be achieved.

Chapter 2

Basic Theoretical Concepts

In this chapter, we provide the basic notions and some theory in order to be able to tackle the indicated classification problem for PDEs of order two in one dependent and two independent variables. In Section 2.1 ("Symmetries of ODEs") we present the theory for ordinary differential equations. The concepts and methods outlined there may be extended to the case of partial differential equations, which is sketched in Section 2.2 ("Symmetries of PDEs"). The remaining three sections, Section 2.3 – 2.5, introduce notions used in Chapters 3, 4 and 5 to structure the presentation of space groups and their invariants. For a further treatment of the subject symmetry groups and differential equations any of the modern textbooks [2, 3, 9, 10] may be consulted. Text books with main emphasis on applications in physics are [11, 12].

2.1 Symmetries of ODEs

As mentioned above, the starting point in our strategy is a listing of groups whose differential invariants determine the general form of a DE that may be invariant under the respective group. We clarify the notion of transformation groups of differential equations in Subsection 2.1.1 ("Transformation Groups of Differential Equations"). The notion of invariance under such a group is investigated in Subsection 2.1.2 ("Infinitesimal Generators and Prolongations"). In Subsection 2.1.3 ("Differential Invariants of Point Transformations"), we introduce the differential invariants for a given transformation group. They are determined by a system of PDEs whose solutions are the desired invariants. Finally, in Subsection 2.1.4 ("Differential Invariants of ODEs: An Example"), we give an example that demonstrates the computation of differential invariants for one chosen plane group.

2.1.1 Transformation Groups of Differential Equations

Introducing new variables into a given DE is a widely used method in order to facilitate the solution process. Usually this is done in an ad hoc manner without guaranteed success. In particular, there is no criterion to decide whether a certain class of transformations will lead to an integrable equation or not. A critical examination of these methods was the starting point for Lie's symmetry analysis. We will now have a look on the behavior of DEs under special kind of transformations.

Let an ODE of order n be given as

$$\omega(x, y, y', \dots, y^{(n)}) = 0. \tag{2.1}$$

The general solution of such an equation is a set of curves in the x-y-plane depending on n parameters $C_1, \dots, C_n$, given by

$$\Theta(x, y, C_1, \dots, C_n) = 0, \tag{2.2}$$

compare any textbook on ordinary DEs, e.g. [3]. Invertible analytic transformations between two planes with coordinates (x, y) and (u, v), respectively, that are of the form

$$u = \sigma(x, y),\ v = \rho(x, y), \tag{2.3}$$

are called *point transformations*. We will encounter them in the form of *one-parameter groups of point transformations*

$$u = \sigma(x, y, a),\ \ v = \rho(x, y, a). \tag{2.4}$$

Here the real parameter a ranges over an open interval including 0, such that for any fixed value of a, equation (2.4) represents a point transformation. In addition, there exists a real group composition Φ such that

$$\bar{x} = \sigma(x, y, a),\ \bar{y} = \rho(x, y, a),\ \bar{\bar{x}} = \sigma(\bar{x}, \bar{y}, \bar{a}),\ \bar{\bar{y}} = \rho(\bar{x}, \bar{y}, \bar{a})$$
$$\Rightarrow \bar{\bar{x}} = \sigma(x, y, \Phi(a, \bar{a})),\ \bar{\bar{y}} = \rho(x, y, \Phi(a, \bar{a})).$$

Group transformations of this kind may be reparametrized such that we have $\Phi(a, \bar{a}) = a + \bar{a}$, and such that $a = 0$ represents the identity element.

An equation (2.1) is said to be *invariant* under the change of variables

$$x = \Phi(u, v),\ y = \Psi(u, v),$$

where $v \equiv v(u)$, if it retains its form under this transformation, i.e. if the functional dependence of the transformed equation on u and v is the same as in the original equation (2.1). Such a transformation is called a *symmetry* of the DE. The same transformation acts on the curves (2.2). If it is a symmetry, the functional dependence of the transformed curves of u and v must be the same as in (2.2). This is not necessarily true for the parameters $C_1, \dots, C_n$ because they do not occur in the DE itself. This means, the entirety of curves described by the two equations is the same, to any fixed values for the constants however may correspond a different curve in either set. In other words the solution curves are permuted among themselves by a symmetry transformation. It is fairly obvious that all symmetry transformations of a given DE form a group, the *symmetry group* of that equation.

2.1.2 Infinitesimal Generators and Prolongations

Let a curve in the (x-y)-plane described by $y = f(x)$ be transformed under a point transformation of the form (2.3) into $v = g(u)$. Now the question arises how the derivative $y' = \frac{df}{dx}$ corresponds to $v' = \frac{dg}{du}$ under this transformation. A simple calculation leads to the *first prolongation*

$$v' = \frac{dv}{du} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'} \equiv \chi(x, y, y').$$

Note that the knowledge of (x, y, y') and the equations of the point transformation (2.3) already determine v' uniquely, the knowledge of the equation of the curve is not required. This may be expressed by saying that the line element (x, y, y') is transformed into the line element (u, v, v') under the action of a point transformation. Similarly, the transformation law for derivatives of second order is obtained as

$$v'' = \frac{dv'}{du} = \frac{\chi_x + \chi_y y' + \chi_{y'} y''}{\sigma_x + \sigma_y y'}.$$

For later applications it would be useful to express the second derivative v'' explicitly in terms of σ and ρ. We do not give this quite lengthy formula here, but instead provide the prolongation formulas for one-parameter groups of point transformations of the form

$$u = \sigma(x, y, a),\ v = \rho(x, y, a). \tag{2.5}$$

Here the transformation properties of the derivatives may be expressed in terms of the prolongation of the corresponding *infinitesimal generator*

$$U = \xi(x, y)\partial_x + \eta(x, y)\partial_y, \tag{2.6}$$

where

$$\xi(x, y) = \frac{d}{da}\sigma(x, y, a)_{|a=0}, \quad \eta(x, y) = \frac{d}{da}\rho(x, y, a)_{|a=0}.$$

The n-th prolongation of (2.6) is now defined as

$$\begin{aligned} U^{(n)} &= U + \sum_{k=1}^{n} \zeta^{(k)} \partial_{y^{(k)}}, \text{ where} \\ \zeta^{(1)} &= D(\eta) - y' D(\xi), \\ \zeta^{(k)} &= D(\zeta^{(k-1)}) - y^{(k)} D(\xi) \quad \text{for } k = 2, 3, \ldots \end{aligned}$$

Hereby, D is the operator of total differentiation w.r.t. x, i.e.

$$D = \partial_x + \sum_{k=1}^{\infty} y^{(k)} \partial_{y^{(k-1)}}.$$

We give the two lowest $\zeta's$ explicitly:

$$\begin{aligned} \zeta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\ \zeta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &\quad - \xi_{yy} y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y''. \end{aligned}$$

These two innocent looking expressions should not divert from the fact that the number of terms in $\zeta^{(k)}$ grows roughly as 2^k. But $\zeta^{(k)}$ is at least linear and homogeneous in $\xi(x, y)$ and $\eta(x, y)$ and its derivatives up to order k. In addition, $\zeta^{(k)}$ does not depend explicitly on x and y but only on $y', y'', \ldots, y^{(k)}$. For $k > 1$, $y^{(k)}$ occurs linearly and y' occurs with power $k + 1$ in $\zeta^{(k)}$.

2.1.3 Differential Invariants of Point Transformations

In order to obtain a classification of possible symmetries of DEs, the invariants of all finite groups of the plane have been determined by Lie. The starting point for this classification was a listing of all finite groups of point transformations of the plane (compare Section 3.2 "Lie's Classification of the Groups of the Plane").

Any r-parameter Lie group may be represented by r infinitesimal generators

$$\xi_i \partial_x + \eta_i \partial_y, \quad i = 1, \ldots, r. \tag{2.7}$$

Any ordinary DE of order m with an r-parameter Lie group as symmetry group has to vanish under all m-th prolongations of the generators (2.7) and vice versa, i.e. this DE is a solution of the following system of linear homogeneous first order partial differential equations:

$$(\xi_i \partial_x + \eta_i \partial_y + \sum_{j=1}^{m} \zeta_i^{(j)} \partial_{y^{(j)}})\Phi = 0, \quad i = 1, \ldots, r, \tag{2.8}$$

where $\Phi \equiv \Phi(x, y, y', y'', \ldots)$. The system (2.8) is called *system of differential invariants*, its fundamental solutions are called the *differential invariants* of the respective Lie group. Lie has discussed these systems in detail, for a recent presentation see [10].

The group property guarantees that (2.8) is a complete system for Φ with $m + r - 2$ solutions. It may be brought into Jacobian normal form, an analogon of the triangular form for matrices, before attempting to solve it. The dependencies of the fundamental solutions may then be chosen such that

$$\begin{aligned} \Phi_1 &\equiv \Phi_1(x, y, y', \ldots, y^{(r-1)}), \\ \Phi_2 &\equiv \Phi_2(x, y, y', \ldots, y^{(r)}), \\ &\vdots \\ \Phi_{m-r+2} &\equiv \Phi_{m-r+2}(x, y, y', \ldots, y^{(m)}). \end{aligned}$$

The invariants are linear in the highest derivative.

2.1.4 Differential Invariants of ODEs: An Example

In this subsection we give an example demonstrating the notions introduced in this section. We consider a group taken from Lie's listing of all finite groups of the plane, namely

$$g_{13} = \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y\}.$$

Prolongation of its three generators up to the third order yields the following system of differential invariants (2.8):

$$\begin{gathered} \Phi_x = 0, \\ 2x\Phi_x + y\Phi_y - y'\Phi_{y'} - 3y''\Phi_{y''} - 5y'''\Phi_{y'''} = 0, \\ x^2\Phi_x + xy\Phi_y - (y'x - y)\Phi_{y'} - 3y''x\Phi_{y''} - (5y'''x + 3y'')\Phi_{y''} = 0. \end{gathered}$$

Using some strategy for solving systems of linear PDEs, for example iterated *narrowing transformations* as introduced in the next section, we might arrive at the following two fundamental solutions:

$$\Phi_1 \equiv y''y^3, \;\; \Phi_2 \equiv y'''y^5 + 3y''y'y^4.$$

The DEs of order not higher than three that have the respective Lie group g_{13} as symmetry group have the general form $\omega(\Phi_1, \Phi_2)$.

2.2 Symmetries of PDEs

The classification problem for partial differential equations has not yet been dealt with. In this thesis partial differential equations of order two in one dependent and two independent variables will be treated.

To this end, we have to start with a listing of continuous space groups, acting on the variables x, y, and z. Finding the differential invariants is accomplished in analogy to the ordinary case: the group generators have to be prolongated to order two; the prolongations are then interpreted as a system of linear PDEs whose fundamental solutions provide a basis of differential invariants.

The prolongation formulas that have to be applied in the case of three variables, one of them dependent on the two others, are given in Subsection 2.2.1 ("Extended Infinitesimal Transformations"). For solving systems of linear PDEs we introduce the *narrowing method* in Subsection 2.2.2 ("Solving Systems of Linear Homogeneous PDEs:").

With these techniques we compute the differential invariants listed in Chapter 4 ("Differential Invariants of Order Two") from the space groups listed in Chapter 3 ("The Space Point Groups"). An example demonstrating this proceeding is presented in Subsection 2.2.3 ("Differential Invariants of PDEs: An Example"), concluding this section.

2.2.1 Extended Infinitesimal Transformations

In this subsection, we introduce the prolongation formulas that apply to the case of partial differential equations, i.e. we deal with one dependent variable u and n independent variables $x = x_1, \ldots, x_n$ (compare [2]). Partial derivatives $\partial x_{i_1} \ldots \partial x_{i_k} u$ are represented by formal variables $u_{i_1 \ldots i_k}$, called *differential indeterminates*. They are symmetric in their indices. The differential variables of order k are denoted by $u^{(k)}$. We also use the convention to sum over the range of multiply occurring indices in products, e.g. $(D_i\xi_j)u_j = \sum_{j=1}^{n}(D_i\xi_j)u_j$.

The one-parameter Lie group of transformations in the parameter ε

$$x_i^* = X_i(x, u; \varepsilon) = x_i + \varepsilon\xi_i(x, u) + O(\varepsilon^2), \tag{2.9}$$

$$u^* = U(x, u; \varepsilon) = u + \varepsilon\eta(x, u) + O(\varepsilon^2), \tag{2.10}$$

$i = 1, 2, \ldots, n$, acting on (x, u)-space has as its infinitesimal generator

$$X = \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u.$$

The k-th extension of (2.9), (2.10), given by

$$x_i^* = X_i(x, u; \varepsilon) = x_i + \varepsilon\xi_i(x, u) + O(\varepsilon^2), \tag{2.11}$$

$$u^* = U(x, u; \varepsilon) = u + \varepsilon\eta(x, u) + O(\varepsilon^2), \tag{2.12}$$

$$\vdots$$

$$u^*_{i_1 i_2 \ldots i_k} = U_{i_1 i_2 \ldots i_k}(x, u, u^{(1)}, \ldots, u^{(k)}; \varepsilon) \tag{2.13}$$

$$= u_{i_1 i_2 \ldots i_k} + \varepsilon\eta^{(k)}_{i_1 i_2 \ldots i_k}(x, u, u^{(1)}, \ldots, u^{(k)}) + O(\varepsilon^2), \tag{2.14}$$

where $i = 1, 2, \ldots, n$ and $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k = 1, 2, \ldots$, has as its k-th extended infinitesimal

$$(\xi(x, u), \eta(x, u), \eta^{(1)}(x, u, u^{(1)}), \ldots, \eta^{(k)}(x, u, u^{(1)}, \ldots, u^{(k)})),$$

with corresponding k-th extended infinitesimal generator

$$X^{(k)} = \xi_i(x,u)\partial_{x_i} + \eta(x,u)\partial_u + \eta_i^{(1)}(x,u,u^{(1)})\partial_{u_i} + \dots \\ + \eta^{(k)}_{i_1 i_2 \dots i_k}\partial_{u_{i_1 i_2 \dots i_k}},$$

$k = 1, 2, \dots$; explicit formulas for the extended infinitesimals $\{\eta^{(k)}\}$ result from the following theorem.

Theorem 2 *The coefficients of the k-th prolongation $X^{(k)}$ may be obtained recursively as follows:*

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \quad i = 1, 2, \dots, n, \tag{2.15}$$

$$\eta^{(k)}_{i_1 i_2 \dots i_k} = D_{i_k}\eta^{(k-1)}_{i_1 i_2 \dots i_k} - (D_{i_k}\xi_j)u_{i_1 i_2 \dots i_{k-1} j}, \tag{2.16}$$

$i_l = 1, 2, \dots, n$ *for* $l = 1, 2, \dots, k$ *with* $k = 2, 3, \dots$

Proof. Let

$$A = \begin{bmatrix} D_1X_1 & \cdots & D_1X_n \\ \vdots & & \vdots \\ D_nX_1 & \cdots & D_nX_n \end{bmatrix}.$$

Then by (2.11) we have

$$A = \begin{bmatrix} D_1(x_1+\varepsilon\xi_1) & D_1(x_2+\varepsilon\xi_2) & \cdots & D_1(x_n+\varepsilon\xi_n) \\ D_2(x_1+\varepsilon\xi_1) & D_2(x_2+\varepsilon\xi_2) & & D_2(x_n+\varepsilon\xi_n) \\ \vdots & \vdots & & \vdots \\ D_n(x_1+\varepsilon\xi_1) & D_n(x_2+\varepsilon\xi_2) & \dots & D_n(x_n+\varepsilon\xi_n) \end{bmatrix} + O(\varepsilon^2) =$$

$$I + \varepsilon B + O(\varepsilon^2),$$

where I is the $n \times n$ identity matrix and

$$B = \begin{bmatrix} D_1\xi_1 & D_1\xi_2 & \cdots & D_1\xi_n \\ D_2\xi_1 & D_2\xi_2 & & D_2\xi_n \\ \vdots & \vdots & & \vdots \\ D_n\xi_1 & D_n\xi_2 & \cdots & D_n\xi_n \end{bmatrix}. \tag{2.17}$$

Then

$$A^{-1} = I - \varepsilon B + O(\varepsilon^2). \tag{2.18}$$

From

$$\begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_n^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 U \\ D_2 U \\ \vdots \\ D_n U \end{bmatrix},$$

(2.12), (2.13), (2.17) and (2.18) it follows that

$$\begin{bmatrix} u_1 + \varepsilon\eta_1^{(1)} \\ u_2 + \varepsilon\eta_2^{(1)} \\ \vdots \\ u_n + \varepsilon\eta_n^{(1)} \end{bmatrix} = [I - \varepsilon B] \cdot \begin{bmatrix} u_1 + \varepsilon D_1\eta \\ u_2 + \varepsilon D_2\eta \\ \vdots \\ u_n + \varepsilon D_n\eta \end{bmatrix} + O(\varepsilon^2),$$

and thus

$$\begin{bmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \\ \vdots \\ \eta_n^{(1)} \end{bmatrix} = \begin{bmatrix} D_1\eta \\ D_2\eta \\ \vdots \\ D_n\eta \end{bmatrix} - B \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

leading to (2.15). Then from

$$\begin{bmatrix} u^*_{i_1 i_2 \ldots i_{k-1} 1} \\ u^*_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ u^*_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix} = \begin{bmatrix} U_{i_1 i_2 \ldots i_{k-1} 1} \\ U_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ U_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 U_{i_1 i_2 \ldots i_{k-1} 1} \\ D_2 U_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ D_n U_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix},$$

(2.13), (2.14), (2.17) and (2.18) we get

$$\begin{bmatrix} u_{i_1 i_2 \ldots i_{k-1} 1} + \varepsilon \eta^{(k)}_{i_1 i_2 \ldots i_{k-1} 1} \\ u_{i_1 i_2 \ldots i_{k-1} 2} + \varepsilon \eta^{(k)}_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ u_{i_1 i_2 \ldots i_{k-1} n} + \varepsilon \eta^{(k)}_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix} =$$

$$[I - \varepsilon B] \begin{bmatrix} u_{i_1 i_2 \ldots i_{k-1} 1} + \varepsilon D_1 \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} \\ u_{i_1 i_2 \ldots i_{k-1} 2} + \varepsilon D_2 \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} \\ \vdots \\ u_{i_1 i_2 \ldots i_{k-1} n} + \varepsilon D_n \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} \end{bmatrix} + O(\varepsilon^2),$$

and hence

$$\begin{bmatrix} \eta^{(k)}_{i_1 i_2 \ldots i_{k-1} 1} \\ \eta^{(k)}_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ \eta^{(k)}_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix} = \begin{bmatrix} D_1 \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} \\ D_2 \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} \\ \vdots \\ D_n \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} \end{bmatrix} - B \cdot \begin{bmatrix} u_{i_1 i_2 \ldots i_{k-1} 1} \\ u_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ u_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix},$$

$i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k-1$ with $k = 2, 3, \ldots$, leading to (2.16).

Specializing Theorem 2 to the case of one dependent variable and two independent variables x_1 and x_2, we have for the extended one-parameter Lie group of transformations given by

$$\begin{aligned} x_i^* &= X_i(x_1, x_2, u; \varepsilon) = x_i + \varepsilon \xi_i(x_1, x_2, u) + O(\varepsilon^2), \ i = 1, 2, \\ u^* &= U(x_1, x_2, u; \varepsilon) = u + \varepsilon \eta(x_1, x_2, u) + O(\varepsilon^2), \\ u_i^* &= U_i(x_1, x_2, u, u_1, u_2; \varepsilon) = \\ &= U_i + \varepsilon \eta_i^{(1)}(x_1, x_2, u, u_1, u_2) + O(\varepsilon^2), \ i = 1, 2, \\ u_{ij}^* &= U_{ij}(x_1, x_2, u, u_1, u_2, u_{11}, u_{12}, u_{22}; \varepsilon) = \\ &= u_{ij} + \varepsilon \eta_{ij}^{(2)}(x_1, x_2, u, u_1, u_2, u_{11}, u_{12}, u_{22}) + O(\varepsilon^2), \ i, j = 1, 2, \end{aligned}$$

etc., the following extended infinitesimals:

$$\eta_1^{(1)} = \frac{\partial \eta}{\partial x_1} + [\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1}] u_1 - \frac{\partial \xi_2}{\partial x_1} u_1 - \frac{\partial \xi_1}{\partial u} (u_1)^2 - \frac{\partial \xi_2}{\partial u} u_1 u_2,$$

$$\eta_2^{(1)} = \frac{\partial \eta}{\partial x_2} + [\frac{\partial \eta}{\partial u} - \frac{\partial \xi_2}{\partial x_2}]u_2 - \frac{\partial \xi_1}{\partial x_2}u_1 - \frac{\partial \xi_2}{\partial u}(u_2)^2 - \frac{\partial \xi_1}{\partial u}u_1 u_2,$$

$$\begin{aligned}\eta_{11}^{(2)} = {} & \frac{\partial^2 \eta}{\partial x_1^2} + [2\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1^2}]u_1 - \frac{\partial^2 \xi_2}{\partial x_1^2}u_2 + [\frac{\partial \eta}{\partial u} - 2\frac{\delta \xi_1}{\partial x_1}]u_{11} \\ & - 2\frac{\partial \xi_2}{\partial x_1}u_{12} + [\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_1}{\partial x_1 \partial u}](u_1)^2 - 2\frac{\partial^2 \xi_2}{\partial x_1 \partial u}u_1 u_2 \\ & - \frac{\partial^2 \xi_1}{\partial u^2}(u_1)^3 - \frac{\partial^2 \xi_2}{\partial u^2}(u_1)^2 u_2 - 3\frac{\partial \xi_1}{\partial u}u_1 u_{11} - \frac{\partial \xi_2}{\partial u}u_2 u_{11} \\ & - 2\frac{\partial \xi_2}{\partial u}u_1 u_{12,}\end{aligned}$$

$$\begin{aligned}\eta_{12}^{(2)} & = \eta_{21}^{(2)} \\ & = \frac{\partial^2 \eta}{\partial x_1 \partial x_2} + [\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_2}{\partial x_1 \partial x_2}]u_2 + [\frac{\partial^2 \eta}{\partial x_2 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1 \partial x_2}]u_1 \\ & - \frac{\partial \xi_2}{\partial x_1}u_{22} + [\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_2}{\partial x_2}]u_{12} - \frac{\partial \xi_1}{\partial x_2}u_{11} - \frac{\partial^2 \xi_2}{\partial x_1 \partial u}(u_2)^2 \\ & + [\frac{\partial^2 \eta}{\partial u^2} - \frac{\partial^2 \xi_1}{\partial x_1 \partial u} - \frac{\partial^2 \xi_2}{\partial x_2 \partial u}]u_1 u_2 - \frac{\partial^2 \xi_1}{\partial x_2 \partial u}(u_1)^2 - \frac{\partial^2 \xi_2}{\partial u^2}u_1(u_2)^2 \\ & - \frac{\partial^2 \xi_1}{\partial u^2}(u_1)^2 u_2 - 2\frac{\partial \xi_2}{\partial u}u_2 u_{12} - 2\frac{\partial \xi_1}{\partial u}u_1 u_{12} - \frac{\partial \xi_1}{\partial u}u_2 u_{11} - \frac{\partial \xi_2}{\partial u}u_1 u_{22},\end{aligned}$$

$$\begin{aligned}\eta_{22}^{(2)} = {} & \frac{\partial^2 \eta}{\partial x_2^2} + [2\frac{\partial^2 \eta}{\partial x_2 \partial u} - \frac{\partial^2 \xi_2}{\partial x_2^2}]u_2 - \frac{\partial^2 \xi_1}{\partial x_2^2}u_1 + [\frac{\partial \eta}{\partial u} - 2\frac{\delta \xi_2}{\partial x_2}]u_{22} \\ & - 2\frac{\partial \xi_1}{\partial x_2}u_{12} + [\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_2}{\partial x_2 \partial u}](u_2)^2 - 2\frac{\partial^2 \xi_1}{\partial x_2 \partial u}u_1 u_2 \\ & - \frac{\partial^2 \xi_2}{\partial u^2}(u_2)^3 - \frac{\partial^2 \xi_1}{\partial u^2}u_1(u_2)^2 - 3\frac{\partial \xi_2}{\partial u}u_2 u_{22} - \frac{\partial \xi_1}{\partial u}u_1 u_{22} - 2\frac{\partial \xi_1}{\partial u}u_2 u_{12,}\end{aligned}$$

etc. These expressions are the basis for calculating second order prolongations of possible symmetries for PDEs in one dependent and two independent variables.

2.2.2 Solving Systems of Linear Homogeneous PDEs

Let us consider a system of linear homogeneous PDEs of the form

$$\sum_{\nu=1}^{n} f^{\mu\nu}(\mathbf{x})\frac{\partial z}{\partial x_\nu} = 0, \quad \mu = 1, \dots, m, \tag{2.19}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, and the $f^{\mu\nu}$ are continuous on some domain $\mathcal{G}$. We introduce the so called *narrowing transformation* for solving systems of the form (2.19) by reducing this problem to solving single linear homogeneous PDEs, compare [5].

More precisely, assume that we know $n-1$ functionally independent solutions $\Psi^1(\mathbf{x}), \dots, \Psi^{n-1}(\mathbf{x})$ for one of the differential equations in (2.19), w.l.o.g. for the m-th equation. Then we have for an arbitrary continuously differentiable function $\zeta(y_1, \dots, y_{n-1})$ that

$$\zeta(\Psi^1(\mathbf{x}), \dots, \Psi^{n-1}(\mathbf{x}))$$

forms an integral of the m-th equation, too. One may now try to narrow down the domain of functions ζ in such a way that $\zeta(\Psi^1(\mathbf{x}), \dots, \Psi^{n-1}(\mathbf{x}))$ also satisfies the remaining differential equations of the system (2.19). For this purpose, we plug $z(\mathbf{x}) = \zeta(y_1, \dots, y_{n-1})$ with $y_\nu = \Psi^\nu(\mathbf{x})$ into the system (2.19), thereby trying to get a system of differential equations for $\zeta(y_1, \dots, y_{n-1})$. By using a slightly altered approach, the following facts might be proven [5]:

Theorem 3 *We assume that (2.19) satisfies the integrability conditions in $\mathcal{G}$, in particular the coefficients are continuously differentiable. Let $\Psi^1(\mathbf{x})$,..., $\Psi^{n-1}(\mathbf{x})$ be a fundamental system of twice continuously differentiable functions for the m-th equation with*

$$\frac{\partial(\Psi^1, \dots, \Psi^{n-1})}{\partial(x_1, \dots, x_{n-1})} \neq 0.$$

By the substitution

$$y_1 = \Psi^1(\mathbf{x}), \dots, y_{n-1} = \Psi^{n-1}(\mathbf{x}), \quad y_n = x_n \tag{2.20}$$

the domain $\mathcal{G}$ is transformed to a convex domain $\mathcal{H}$ (if this is not a priori the case, then the domain $\mathcal{G}$ has to be made smaller appropriately). Finally, *we assume that in no subdomain of $\mathcal{G}$ we have $f^{mn} \equiv 0$. Then the integrals of (2.19) are exactly those functions $z(\mathbf{x}) = \zeta(\Psi^1, \dots, \Psi^{n-1})$ where ζ runs through the solutions of the system*

$$\sum_{k=1}^{n-1} g^{\mu k}(\mathbf{x}) \frac{\partial \zeta}{\partial y_k} = 0, \quad \mu = 1, \dots, m-1, \tag{2.21}$$

where the functions

$$g^{\mu k} = \sum_{\nu=1}^{n} f^{\mu\nu} \Psi^k_{x_\nu}, \quad \mu = 1, \dots, m-1,\ k = 1, \dots, n-1,$$

depend only on $y_1, \dots, y_{n-1}$ after carrying out substitution (2.20). In addition, system (2.21) satisfies the integrability conditions.

Further details may be found in [5].

2.2.3 Differential Invariants of PDEs: An Example

We give an example for the computation of a basis of differential invariants of a given Lie group in three space. We assume that we can solve single linear PDEs. The group $\mathbf{ip}_{22} = \{\partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_x + y\partial_y + \partial_z\}$ under consideration is taken from the listing in Section 3.4 ("Lie's Imprimitive Space Groups"). By prolongation up to order two we get the following system of differential invariants:

$$\begin{gathered}
\Phi_x = 0,\ \Phi_y = 0, \\
x\Phi_y - z_y\Phi_{z_x} - z_{yy}\Phi_{z_{xy}} - 2z_{xy}\Phi_{z_{xx}} = 0, \\
x\Phi_x - y\Phi_y - z_x\Phi_{z_x} + z_y\Phi_{z_y} + 2z_{yy}\Phi_{z_{yy}} - 2z_{xx}\Phi_{z_{xx}} = 0, \\
y\Phi_x - z_x\Phi_{z_y} - 2z_{xy}\Phi_{z_{yy}} - z_{xx}\Phi_{z_{xy}} = 0, \\
x\Phi_x + y\Phi_y + \Phi_z - z_x\Phi_{z_x} - z_y\Phi_{z_y} - 2z_{yy}\Phi_{z_{yy}} - 2z_{xy}\Phi_{z_{xy}} - 2z_{xx}\Phi_{z_{xx}} = 0.
\end{gathered}$$

These six equations in eight variables allow two fundamental solutions. The first two equations express that Φ does not explicitly depend on x and y. So we consider the remaining four equations in six variables. To this end, let $[v_1^{(0)}, v_2^{(0)}, v_3^{(0)}, v_4^{(0)}, v_5^{(0)}, v_6^{(0)}] = [z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}]$. An integral basis for the third remaining equation leads to the following narrowing transformation:

$$v_1^{(1)} = v_1^{(0)}, v_2^{(1)} = v_2^{(0)}, v_3^{(1)} = v_4^{(0)}, v_6^{(1)} = v_3^{(0)},$$

$$v_4^{(1)} = \frac{v_5^{(0)} v_2^{(0)} - v_4^{(0)} v_3^{(0)}}{v_2^{(0)}},$$

$$v_5^{(1)} = \frac{v_6^{(0)} (v_2^{(0)})^2 + v_4^{(0)} (v_3^{(0)})^2 - 2 v_3^{(0)} v_5^{(0)} v_2^{(0)}}{(v_2^{(0)})^2}.$$

The other three equations are transformed by it into

$$v_2^{(1)} v_6^{(1)} \frac{\partial \Phi^{(1)}}{v_2^{(1)}} + 2(v_2^{(1)} v_4^{(1)} + v_3^{(1)} v_6^{(1)}) \frac{\partial \Phi^{(1)}}{v_3^{(1)}} + v_2^{(1)} v_5^{(1)} \frac{\partial \Phi^{(1)}}{v_4^{(1)}} - 2 v_6^{(1)} v_5^{(1)} \frac{\partial \Phi^{(1)}}{v_5^{(1)}} = 0,$$

$$- v_2^{(1)} \frac{\partial \Phi^{(1)}}{v_2^{(1)}} - 2 v_3^{(1)} \frac{\partial \Phi^{(1)}}{v_3^{(1)}} + 2 v_5^{(1)} \frac{\partial \Phi^{(1)}}{v_5^{(1)}} = 0,$$

$$\frac{\partial \Phi^{(1)}}{v_1^{(1)}} - v_2^{(1)} \frac{\partial \Phi^{(1)}}{v_2^{(1)}} - 2 v_3^{(1)} \frac{\partial \Phi^{(1)}}{v_3^{(1)}} - 2 v_4^{(1)} \frac{\partial \Phi^{(1)}}{v_4^{(1)}} - 2 v_5^{(1)} \frac{\partial \Phi^{(1)}}{v_5^{(1)}} = 0.$$

Now we iterate. An integral basis for the second of the above equations leads to the narrowing transformation

$$v_1^{(2)} = v_1^{(1)}, v_3^{(2)} = v_4^{(1)}, v_5^{(2)} = v_2^{(1)},$$

$$v_2^{(2)} = \frac{v_3^{(1)}}{(v_2^{(1)})^2}, \; v_4^{(2)} = v_5^{(1)} (v_2^{(1)})^2.$$

The other two equations are transformed by it into

$$2 v_3^{(2)} \frac{\partial \Phi^{(2)}}{v_2^{(2)}} + v_4^{(2)} \frac{\partial \Phi^{(2)}}{v_3^{(2)}} = 0,$$

$$\frac{\partial \Phi^{(2)}}{v_1^{(2)}} - 2 v_3^{(2)} \frac{\partial \Phi^{(2)}}{v_3^{(2)}} - 4 v_4^{(2)} \frac{\partial \Phi^{(2)}}{v_4^{(2)}} = 0.$$

We iterate again; an integral basis for the first of the above equations leads to the narrowing transformation

$$v_1^{(3)} = v_1^{(2)}, v_2^{(3)} = v_4^{(2)}, v_4^{(3)} = v_2^{(2)},$$

$$v_3^{(3)} = -v_2^{(2)} v_4^{(2)} + (v_3^{(2)})^2.$$

The other equation is transformed by it into

$$\frac{\partial\Phi^{(3)}}{v_1^{(3)}} - 4v_2^{(3)}\frac{\partial\Phi^{(3)}}{v_2^{(3)}} - 4v_3^{(3)}\frac{\partial\Phi^{(3)}}{v_3^{(3)}} = 0.$$

An integral basis for this single linear PDE is

$$\Phi_1^{(3)} = v_2^{(3)}e^{4v_1^{(3)}},\ \Phi_2^{(3)} = v_3^{(3)}e^{4v_1^{(3)}}.$$

By inverting the above three narrowing transformations we arrive at two fundamental solutions of the original system of differential invariants. They are

$$\Phi_1 = (z_{xy}^2 - z_{xx}z_{yy})e^{4z}, \quad \Phi_2 = (z_{yy}z_x^2 - 2z_xz_yz_{xy} + z_{xx}z_y^2)e^{4z}.$$

Then $\omega(\Phi_1, \Phi_2) = 0$ represents the most general PDE of order two in one dependent and two independent variables which has $\mathbf{ip}_{22}$ as symmetry group.

We show for example that Φ_1 is invariant under ∂_x and $y\partial_y$. We have that ∂_x is the infinitesimal generator of the one-parameter transformation group

$$\bar{x} = x + a,\ \bar{y} = y,\ \bar{z} = z. \tag{2.22}$$

Now,

$$\bar{\Phi}_1 = (\bar{z}_{\bar{x}\bar{y}}^2 - \bar{z}_{\bar{x}\bar{x}}\bar{z}_{\bar{y}\bar{y}})e^{4\bar{z}} = (z_{\bar{x}y}^2 - z_{\bar{x}\bar{x}}z_{yy})e^{4z} = (z_{xy}^2 - z_{xx}z_{yy})e^{4z} = \Phi_1,$$

i.e. Φ_1 is invariant under (2.22). Note that the third equality sign holds because of $z_{\bar{x}} = z_x\frac{d\bar{x}}{dx} = z_x$.

On the other hand, $y\partial_y$ is the infinitesimal generator of the one-parameter group of transformations

$$\bar{x} = x + ay,\ \bar{y} = y,\ \bar{z} = z. \tag{2.23}$$

Again, we have

$$\bar{\Phi}_1 = (\bar{z}_{\bar{x}\bar{y}}^2 - \bar{z}_{\bar{x}\bar{x}}\bar{z}_{\bar{y}\bar{y}})e^{4\bar{z}} = (z_{\bar{x}y}^2 - z_{\bar{x}\bar{x}}z_{yy})e^{4z} = (z_{xy}^2 - z_{xx}z_{yy})e^{4z} = \Phi_1,$$

i.e. Φ_1 is invariant under (2.23). For the third equality sign, note that

$$z_{\bar{x}} = z_x\frac{d\bar{x}}{dx} + z_y\frac{d\bar{x}}{dy} = z_x + az_y,$$

and hence

$$z_{\bar{x}y}^2 = (z_{xy} + az_{yy})^2 = z_{xy}^2 + 2az_{xy}z_{yy} + a^2 z_{yy}^2.$$

Furthermore,

$$\begin{aligned} z_{\bar{x}\bar{x}} &= (z_x + az_y)_{\bar{x}} = z_{x\bar{x}} + az_{y\bar{x}} \\ &= (z_{xx}\frac{d\bar{x}}{dx} + z_{xy}\frac{d\bar{x}}{dy}) + a(z_{yx}\frac{d\bar{x}}{dx} + z_{yy}\frac{d\bar{x}}{dy}) \\ &= z_{xx} + az_{xy} + az_{xy} + a^2 z_{yy} \\ &= z_{xx} + 2az_{xy} + a^2 z_{yy}. \end{aligned}$$

Hence we have

$$\begin{aligned} (z_{\bar{x}y}^2 - z_{\bar{x}\bar{x}}z_{yy}) &= (z_{xy}^2 + 2az_{xy}z_{yy} + a^2 z_{yy}^2) - (z_{xx}z_{yy} + 2az_{xy}z_{yy} + a^2 z_{yy}^2) \\ &= z_{xy}^2 - z_{xx}z_{yy}. \end{aligned}$$

2.3 Two Types of Invariants

In this section we introduce the notion of a basis of differential invariants. In the formulation, we specialize to the order two case with one dependent variable z and two independent variables x, y. Additionally, the concept of lower invariants is introduced.

We denote the variables by $V := \{x, y, z\}$ and the differential variables by $W := \{z_x, z_y, z_{xx}, z_{xy}, z_{yy}\}$. Their union is denoted by $U := V \cup W$. Let G_r be an r-parameter space point transformation group with generators

$$X_\alpha = \sum_{v \in V} \xi_{\alpha,v}(V)\partial_v, \qquad \alpha = 1, \dots, r.$$

A function $F(U)$ is an invariant of the two times extended transformation group with generators

$$X_\alpha^{(2)} = \sum_{v \in V} \xi_{\alpha,v}(V)\partial_v + \sum_{w \in W} \zeta_{\alpha,w}(V)\partial_w, \qquad \alpha = 1, \dots, r,$$

if and only if it solves the following system of homogeneous linear partial differential equations:

$$X_\alpha^{(2)}(F) \equiv 0, \qquad \alpha = 1, \dots, r. \tag{2.24}$$

Hereby, $\zeta_{\alpha,w}$ for $w \in W$ are the prolongations of order one and two, respectively. The function F is called a *differential invariant of order two.*

We define r_* to be the rank of the coefficient matrix of the system (2.24), i.e.

$$r_* := rank([\xi_{\alpha,v}, \zeta_{\alpha,w}]_{\alpha=1,\dots,r}^{v\in V,\, w\in W}).$$

The number of variables U involved in (2.24) is eight, so any $n := 8 - r_*$ functionally independent solutions $\Psi_1(U), \dots, \Psi_n(U)$ of (2.24) form a basis of its solution space, i.e. any other solution of (2.24) has the form

$$F = \Phi(\Psi_1(U), \dots, \Psi_n(U)).$$

Since the solutions of (2.24) are by definition the differential invariants, $\Psi_1(U), \dots, \Psi_n(U)$ are called a *basis of differential invariants* for G_r. In the following chapters, the totality of differential invariants for a given group will be represented by a basis of differential invariants for the group.

We will always be able to provide a basis of differential invariant for any r-parameter group with $r \leq 7$. For groups with more than seven parameters, the rank r_* in general is eight, hence the system (2.24) admits only trivial constant solutions. Only in exceptional cases, the rank r_* might be less than eight, thus allowing to compute an invariant basis. These exceptions are also documented in this work.

Let us now assume that G_r is an r-parameter group with $r \geq 8$ and $r_* = 8$. We might replace the system (2.24) with an equivalent system by choosing eight equations that leave the new system with full rank, w.l.o.g. we consider

$$X_\alpha^{(2)}(F) \equiv 0, \qquad \alpha = 1, \dots, 8. \tag{2.25}$$

We can expect to find a non-trivial solution of (2.25), if the determinant of its quadratic coefficient matrix

$$d := \det([\xi_{\alpha,v}, \zeta_{\alpha,w}]_{\alpha=1,\dots,8}^{v\in V,\, w\in W})$$

vanishes. We call any irreducible factor f of d that satisfies

$$X_\alpha^{(2)}(f) \equiv_f 0, \qquad \alpha = 1, \dots, 8, \tag{2.26}$$

a *lower invariant* of G_r. The computation proceeds by factoring the determinant d into irreducible factors and applying the test (2.26) for each factor.

2.4 Basic Notions for Lie Algebras

In this section we introduce several basic notions for Lie algebras of vector fields, the only type of Lie algebras considered in this work. The notions introduced for later use are commutator table, derived series, isomorphism and similarity. Details and the purely algebraic theory of Lie algebras as introduced by W. Killing may be found in the book by Jacobson [4].

Definition: A *Lie algebra of vector fields* is a vector space $\mathbf{L}$ of operators $X = \sum_i \xi_i(x)\partial_{x_i}$ endowed with the *commutator* $[\cdot, \cdot]$ such that

$$[X, Y] := XY - YX \in \mathbf{L}, \qquad X, Y \in \mathbf{L}.$$

Remark: The commutator is bilinear, skew-symmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \qquad X, Y, Z \in \mathbf{L}.$$

From now on, we denote a Lie algebra of vector fields by LA for short.

Definition: We say that the finite LA $\mathbf{L}$ has *dimension* r, written $\dim(\mathbf{L}) = r$, if $\mathbf{L}$ is the linear span of r linearly independent operators $X_1, \dots, X_r$ with constant coefficients, written $\mathbf{L} = \{X_1, \dots, X_r\}$. We call $X_1, \dots, X_r$ a *basis* of $\mathbf{L}$. The matrix

$$[[X_i, X_j]]_{i=1,\dots,r}^{j=1,\dots,r}$$

is called the *commutator table* of $\mathbf{L}$ w.r.t. $X_1, \dots, X_r$. The constants $c_{i,j,k}$ in the relations

$$[X_i, X_j] = \sum_{k=1}^{r} c_{i,j,k} X_k, \qquad i,\ j = 1, \dots, r$$

are called *structure constants*.

The commutator table is skew symmetric and has only zeros in the diagonal.

Example: Let $\mathbf{L} = \{X_1, X_2, X_3\}$, where $X_i = x^{i-1}\partial_x$ for $i = 1, 2, 3$. The commutator table of $\mathbf{L}$ is

$$\begin{bmatrix} 0 & X_1 & 2X_2 \\ -X_1 & 0 & X_3 \\ -2X_2 & -X_3 & 0 \end{bmatrix}.$$

Definition: The *derived algebra* $\mathbf{L}^{(1)}$ of a LA $\mathbf{L}$ is the LA generated by all commutators of $\mathbf{L}$

$$\mathbf{L}^{(1)} = [\mathbf{L}, \mathbf{L}] = \{[X, Y] \mid X, Y \in \mathbf{L}\}.$$

Derived algebras of higher order are defined recursively by

$$\mathbf{L}^{(i+1)} = (\mathbf{L}^{(i)})^{(1)}, \qquad i \geq 1.$$

The *derived series* of a finite LA is the sequence of dimensions of its derived algebras. We present the derived series in the form of the finite sequence

$$(\dim(\mathbf{L}), \dim(\mathbf{L}^{(1)}), \ldots, \dim(\mathbf{L}^{(t)})),$$

where t is the smallest number $0 \leq t \leq \dim(\mathbf{L}) + 1$ such that

$$\mathbf{L}^{(t)} = \mathbf{L}^{(t+i)} \qquad \forall i \geq 1.$$

Example: Consider $\mathbf{L} = \mathbf{ip}_{13,A} = \{\partial_x, \partial_y, y\partial_y + \partial_z\}$ with $\dim(\mathbf{L}) = 3$. By considering the commutator table of $\mathbf{L}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \partial_y \\ 0 & -\partial_y & 0 \end{bmatrix},$$

we see that $\mathbf{L}^{(1)} = [\mathbf{L}, \mathbf{L}] = \{\partial_y\}$, and hence $\dim(\mathbf{L}^{(1)}) = 1$. Since $\mathbf{L}^{(2)} = [\mathbf{L}^{(1)}, \mathbf{L}^{(1)}] = \{\}$, we have $\dim(\mathbf{L}^{(2)}) = 0$. Hence the derived series is

$$(\dim(\mathbf{L}), \dim(\mathbf{L}^{(1)}), \dim(\mathbf{L}^{(2)})) = (3, 1, 0).$$

The group classification of ordinary DEs $f(x, y, y', \ldots, y^{(m)}) = 0$ in Section 4.1 is based on the enumeration of all possible LAs (infinitesimal groups in Lie's terminology) in the (x, y)-plane. In the enumeration given in Section 3.2, the algebras are maximally simplified by a proper choice of bases and by means of a suitable change of variables. Associated with these two types of simplifying transformations are two distinctly different notions: isomorphic and similar LAs.

Definition (Isomorphic LAs) Let $\mathbf{L}$ and $\mathbf{K}$ be two LAs, and let $\dim(\mathbf{L}) = \dim(\mathbf{K})$. A linear one-to-one map f of $\mathbf{L}$ onto $\mathbf{K}$ is called an *isomorphism* if it preserves the commutation relation, i.e. if

$$f([X, Y]) = [f(X), f(Y)], \qquad X, Y \in \mathbf{L}.$$

If the LAs $\mathbf{L}$ and $\mathbf{K}$ can be related by an isomorphism, they are termed *isomorphic* LAs.

Theorem Two finite-dimensional LAs are isomorphic if and only if one can choose bases for the algebras such that the algebras have, in these bases, equal structure constants, i.e. the same table of commutators.

Definition (Similarity) The LAs of vector fields $\mathbf{L}$ and $\overline{\mathbf{L}}$ are *similar* if one is obtained from the other by a change of variables. It means that the operators $X = \sum \xi_i(x)\partial_{x_i}$ and $\overline{X} = \sum \bar{\xi}_i(\bar{x})\partial_{\bar{x}_i}$ of $\mathbf{L}$ and $\overline{\mathbf{L}}$ are related by

$$\bar{x}_i = \bar{x}_i(x), \quad \bar{\xi}_i = X(\bar{x}_i)|_{x=\bar{x}^{-1}(x)}, \qquad i = 1, \dots, n,$$

where $\bar{x}^{-1}(x)$ denotes the inverse of the change of variables $\bar{x}(x)$.

Example: Let $X = \partial_y$ be the operator of translation in y in the (x, y)-plane. We compute the transformed operator $\overline{X}$ under the change of variables $\bar{x} = x + y$, $\bar{y} = x - y$. Without having to consider the inverse variable change $x = (\bar{x} + \bar{y})/2$, $y = (\bar{x} - \bar{y})/2$ we get

$$\bar{\xi}_x = \partial_y(\bar{x})|_{x=\bar{x}^{-1}} = 1, \ \bar{\xi}_y = \partial_y(\bar{y})|_{x=\bar{x}^{-1}} = -1, \ \text{i.e. } \overline{X} = \partial_{\bar{x}} - \partial_{\bar{y}}.$$

Remark In order that two LAs with the same dimension and the same number of variables are similar, it is necessary that they are isomorphic. The converse is not true.

It is precisely similarity that is of use in group analysis as a criterion of reducibility of one DE to another by a suitable change of variables. Nonetheless, establishing isomorphism is important as a first step for the determination of similarity. We state a theorem about deciding similarity for computational purposes; we introduce the notion of connectedness first.

Definition (Connectedness) Differential operators $X_1, \dots, X_r$ are said to be *connected* if there exist functions $\lambda_i(x)$, not all zero, such that

$$\lambda_1(x)X_1 + \ldots + \lambda_r(x)X_r = 0, \tag{2.27}$$

this being satisfied as an operator identity in a neighborhood of a generic x. If the relation (2.27) implies $\lambda_1 = \ldots = \lambda_r = 0$, we say that the operators $X_1, \dots, X_r$ are *unconnected.*

Theorem (Similarity of groups) In order that two groups $X_1, \dots, X_n$ in the (x, y, z)-space and $\overline{X}_1, \dots, \overline{X}_n$ in the $(\overline{x}, \overline{y}, \overline{z})$-space, respectively, are similar, the following conditions are necessary and sufficient.

1. The two groups must have the same structure, i.e. if the relations $(X_i, X_j) = \sum_{k=1}^{r} c_{ijk} X_k$ are valid, it must be possible to choose suitable linear combinations $\overline{\overline{X}}_i = \sum_{j=1}^{r} \gamma_{ij} \overline{X}_j$ with constants γ_{ij} such that the $\overline{\overline{X}}_i$ satisfy the relations $(\overline{\overline{X}}_i, \overline{\overline{X}}_j) = \sum_{k=1}^{r} c_{ijk} \overline{\overline{X}}_k$, i.e. the structure constants have the same values as for the X_i.

2. If $X_1, \ldots, X_n$ are unconnected whereas $X_{n+i} = \sum_{j=1}^{n} \Phi_{ij}(x, y, z) X_j$, the corresponding generators $\overline{X}_1, \ldots, \overline{X}_n$ are also unconnected, whereas the relations $\overline{X}_{n+i} = \sum_{j=1}^{n} \overline{\Phi}_{ij}(\overline{x}, \overline{y}, \overline{z}) \overline{X}_j$ are valid such that the equations $\Phi_{ij}(x, y, z) = \overline{\Phi}_{ij}(\overline{x}, \overline{y}, \overline{z})$ are consistent. In particular, they should not generate a relation among x, y and z on the one hand, or $\overline{x}$, $\overline{y}$ and $\overline{z}$ on the other.

The proof of this theorem may be found in [3].

2.5 Systems of Imprimitivity

In this section we introduce systems of imprimitivity. Lie used their number and type to obtain a classification of groups allowed by various manifolds [7, 8]. The presentation of plane and space groups in the following chapters is organized that way. Transitivity and primitivity are notions from substitution theory; Lie extended them to transformation groups. Details may be found in [7, 10].

We introduce the notion of transitivity first.

Definition (Transitive and Intransitive Groups) Let D be a domain in $\mathbb{R}^n$. An r-parameter transformation group $T_a : D \to D$ with parameter space $P \subseteq \mathbb{R}^r$ is called *transitive* iff

$$\forall \bar{x}, \bar{y} \in D \; \exists \bar{a} \in P : T_{\bar{a}}(\bar{x}) = \bar{y}. \tag{2.28}$$

If (2.28) is not satisfied, we call T *intransitive.*

Example: The translation group $T_{(a_1,a_2)}(x_1, x_2) = (x_1 + a_1, x_2 + a_2)$ of the plane is obviously transitive. For every choice of two points $(\bar{x}_1, \bar{x}_2)$, $(\bar{y}_1, \bar{y}_2)$ we have $T_{(\bar{y}_1 - \bar{x}_1, \bar{y}_2 - \bar{x}_2)}(\bar{x}_1, \bar{x}_2) = (\bar{y}_1, \bar{y}_2)$. The transformation group $T_{(a_1,a_2)}(x_1, x_2) = (x_1, x_2 + a_1 x_1 + a_2)$ is obviously intransitive. For every choice of two points $(\bar{x}_1, \bar{x}_2)$, $(\bar{y}_1, \bar{y}_2)$ with $\bar{x}_1 \neq \bar{y}_1$, there is no choice of parameters $(\bar{a}_1, \bar{a}_2)$ such that $T_{(\bar{a}_1, \bar{a}_2)}$ maps the first point into the latter.

If the infinitesimal generators of the transformation group are known, the following criterion may be used to decide transitivity.

Theorem: A group of the n-dimensional space with r infinitesimal generators $X_k = \sum_{i=1}^{n} \xi_{k,i}(x)\partial_{x_i}$, $k = 1, \ldots, r$, is transitive iff

$$rank(\left[\xi_{k,i}\right]_{k=1,\ldots,n}^{i=1,\ldots,r}) = n. \tag{2.29}$$

Hereby, *rank* denotes the maximal number of unconnected lines of the matrix.

Example: We consider $\mathbf{p}_6 = \{\partial_x, \partial_y, \partial_z, x\partial_y - y\partial_x, x\partial_z - z\partial_x, y\partial_z - z\partial_y\}$, a space group from Section 3.3 ("The Primitive Space Groups"). By applying criterion (2.29)

$$rank\left(\begin{bmatrix} 1 & 0 & 0 & y & 0 & -z \\ 0 & 1 & 0 & -x & z & 0 \\ 0 & 0 & 1 & 0 & -y & x \end{bmatrix}^T\right) = 3$$

we conclude that $\mathbf{p}_6$ is transitive. As a second example, we consider the space group $\mathbf{ip}_1 = \{\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, y^2\partial_y + xy\partial_x\}$ from Section 3.4 ("The Imprimitive Space Groups"). By applying criterion (2.29)

$$rank\left(\begin{bmatrix} 1 & 0 & x & y & 0 & 0 & x^2 & xy \\ 0 & 1 & 0 & 0 & x & y & xy & y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\right) = 2$$

we conclude that $\mathbf{ip}_1$ is intransitive.

Prior to introducing imprimitive groups, we introduce the notion of an invariant manifold.

Definition (Invariant Manifold) A manifold $\mathcal{M} \subseteq \mathbb{R}^n$ is called *invariant* under the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ iff $T(\mathcal{M}) = \mathcal{M}$. It is invariant under the transformation group T iff it is invariant under all transformations of T.

Definition (Primitive and Imprimitive Groups) A transitive group of the n-dimensional space is called *imprimitive* iff it determines at least one partition of the space into $(n-q)$-parameter families of invariant q-dimensional manifolds for some q with $1 \leq q \leq n-1$. If no such partition exists, the group is called *intransitive.*

Example: The translation group of the plane considered in the example above $T_{(a_1,a_2)}(x_1, x_2) = (x_1 + a_1, x_2 + a_2)$ is transitive and transforms any line $x_2 = \alpha x_1 + \beta$ into $\bar{x}_2 = \alpha \bar{x}_1 + \beta'$ with the same value of α. For fixed α, the one-parameter family of one-dimensional invariant manifolds $\mathcal{F} = \{\mathcal{M}_\beta \mid \beta \in \mathbb{R}\}$, where $\mathcal{M}_\beta = \{(x_1, x_2) \mid x_2 = \alpha x_1 + \beta\}$, is a partition of the plane, hence we have that T is imprimitive.

Chapter 3

The Space Point Groups

Lie used the term *type* of a point group for the full equivalence class of the respective group w.r.t. point transformations. Therefore the type is completely determined by providing a canonical representative for the class. Lie gave a classification of the groups of the plane in [8] by providing a canonical representative for each type.

In this chapter we list all finite continuous space groups. After introducing some notation in the first section, Lie's classification of the groups of the plane is discussed in Section 3.2. The following three sections, Section 3.3-3.5, present listings of the groups of the 3-dimensional space. They are divided into three categories according to their systems of imprimitivity. These three categories are not claimed to be disjoint, but they represent a full list of groups. While the groups in section 3.5 are extracted from a paper by Ugo Amaldi [1] who completed Lie's listing, the remaining sections of this chapter can be extracted from [7, 8].

This listing of space groups is the basis for the classification problem for PDEs in one dependent and two independent variables. We also indicate the group size for any listed group; the only exceptions are groups whose number of generators can be seen immediately, and the Amaldi groups of type B, which have not been processed within the frame of this work.

3.1 Some Notation

In this subsection we introduce some notation used to simplify the presentation of space groups.

Definition: For an integer variable v and an integer constant or variable n we write $v \to n$ for $v = 1, \dots, n$. We write $v \to^* n$ for $v = 0, \dots, n$.

Definition: For a term t, we denote by $t_{v=l}^{u}$ the list

$$t[v = l], t[v = l + 1], \dots, t[v = u].$$

Similarly $t_{v=l,\dots,u}^{\bar{v}=\bar{l},\dots,\bar{u}}$ denotes

$$t[v = l, \bar{v} = \bar{l}], \dots, t[v = l, \bar{v} = \bar{u}], \dots, t[v = u, \bar{v} = \bar{l}], \dots, t[v = u, \bar{v} = \bar{u}].$$

Also several multi-parameter ranges are allowed, for example $t_{i\to m,\ j\to n}^{k\to l}$ etc.

Example:

$$(x^i \partial_y + i x^{i-1} \partial_z)_{i=0}^{s} = \partial_y, x\partial_y + \partial_z, \dots, x^s \partial_y + s x^{s-1} \partial_z.$$

$$(x^i y^j)_{i\to m}^{j\to n} = xy, \dots, xy^n, x^2 y, \dots, x^2 y^n, \dots, x^m y, \dots, x^m y^n.$$

Definition: For a truth value formula b, we denote by $[b]$ the truth value of b w.r.t. its actual arguments.

Example:

$$[i \geq t] = 1 \text{ if } i \geq t,\ [i \geq t] = 0 \text{ otherwise.}$$

Definition: For a generator $g = \xi_1 \partial_x + \xi_2 \partial_y + \xi_3 \partial_z$ and $v \in \{x, y, z\}$ we denote by g_v the generator received by applying the operator ∂_v to the coefficients of g.

Example: Let $g := x^2 \partial_x + sxy\partial_y + [(s-2)zx + sy]\partial_z$. Then

$$g_x = 2x\partial_x + sy\partial_y + (s-2)z\partial_z.$$

3.2 Lie's Classification of the Groups of the Plane

The importance of the plane groups originates from the fact that any group of point symmetries of an ODE must be similar to one of those finite groups of point transformations. Furthermore, the classification of space groups is based on a suitable extension of the groups of the plane. Any of the groups listed below is a representative of a full equivalence class that may be obtained by applying an arbitrary point transformation. Actually, not all of them do occur as symmetry group of an ODE of given order, compare Chapter 4.1 ("Invariants of Lie's Plane Groups").

The listing of plane groups given below is organized by their *systems of imprimitivity.* It is taken from [10] and follows closely the one given by Lie [8] with some minor corrections included.

Primitive Groups

$\mathbf{g}_1 : \{\partial_x, y\partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y\}$.

$\mathbf{g}_2 : \{\partial_x, \partial_y, x\partial_y, y\partial_y, x\partial_x, y\partial_x\}$.

$\mathbf{g}_3 : \{\partial_x, \partial_y, x\partial_y, y\partial_y, x\partial_x, y\partial_x, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y\}$.

Two Systems of Imprimitivity $x = const.$ and $y = const.$

$\mathbf{g}_4 : \{\partial_y, y\partial_y\}$.

$\mathbf{g}_5 : \{\partial_x, \partial_y, y\partial_y\}$.

$\mathbf{g}_6 : \{\partial_x, \partial_y, y\partial_y, x\partial_x\}$.

$\mathbf{g}_7 : \{\partial_x, \partial_y, x\partial_x + cy\partial_y\}$, $c \notin \{0, 1\}$.

$\mathbf{g}_8 : \{\partial_y, y\partial_y, y^2\partial_y\}$.

$\mathbf{g}_9 : \{\partial_x, \partial_y, y\partial_y, y^2\partial_y\}$.

$\mathbf{g}_{10} : \{\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y\}$.

$\mathbf{g}_{11} : \{\partial_x, \partial_y, y\partial_y, x\partial_x, y^2\partial_y\}$.

$\mathbf{g}_{12} : \{\partial_x, \partial_y, y\partial_y, x\partial_x, y^2\partial_y, x^2\partial_x\}$.

System of Imprimitivity $x = const.$

$\mathbf{g}_{13} : \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y\}$.

$\mathbf{g}_{14} : \{y\partial_y, \partial_x, x\partial_x, x^2\partial_x + xy\partial_y\}$.

$\mathbf{g}_{15} : \{\partial_y, x\partial_y, \Phi_1(x)\partial_y, \dots, \Phi_r(x)\partial_y\}$, size: $r + 2 \geq 2$.

$\mathbf{g}_{16} : \{\partial_y, x\partial_y, \Phi_1(x)\partial_y, \dots, \Phi_r(x)\partial_y, y\partial_y\}$, size: $r + 3 \geq 3$.

$\mathbf{g}_{17} : \{e^{a_k x}\partial_y, xe^{a_k x}\partial_y, \dots, x^{\rho_k}e^{a_k x}\partial_y, \partial_x\}$,
$l \geq 1$, $a_1(a_1 - 1) = 0$, size: $1 + l + \sum_{k=1}^{l} \rho_k \geq 3$.

$\mathbf{g}_{18} : \{e^{a_k x}\partial_y, xe^{a_k x}\partial_y, \dots, x^{\rho_k}e^{a_k x}\partial_y, y\partial_y, \partial_x\}$,
$l \geq 1$, $(a_1, a_2) = (0, 1)$, size: $2 + l + \sum_{k=1}^{l} \rho_k \geq 4$.

$\mathbf{g}_{19} : \{\partial_y, x\partial_y, \dots, x^{r-1}\partial_y, \partial_x, x\partial_x + cy\partial_y\}$, size: $r + 2 \geq 4$.

$\mathbf{g}_{20} : \{\partial_y, x\partial_y, \dots, x^{r-1}\partial_y, \partial_x, x\partial_x + (ry + x^r)\partial_y\}$, size: $r + 2 \geq 3$.

$\mathbf{g}_{21} : \{\partial_y, x\partial_y, \dots, x^{r-1}\partial_y, y\partial_y, \partial_x, x\partial_x\}$, size: $r + 3 \geq 5$.

$\mathbf{g}_{22} : \{\partial_y, x\partial_y, \dots, x^{r-1}\partial_y, \partial_x, 2x\partial_x + (r-1)y\partial_y, x^2\partial_x + (r-1)xy\partial_y\}$,
size: $r + 3 \geq 5$.

$\mathbf{g}_{23} : \{\partial_y, x\partial_y, \dots, x^{r-1}\partial_y, y\partial_y, \partial_x, x\partial_x, x^2\partial_x + (r-1)xy\partial_y\}$, size: $r + 4 \geq 6$.

System of Imprimitivity $y = \alpha x + const.$

$\mathbf{g}_{24} : \{\partial_x, \partial_y, x\partial_x + y\partial_y\}$.

$\mathbf{g}_{25} : \{\partial_y, x\partial_x + y\partial_y\}$.

$\mathbf{g}_{26} : \{\partial_x, \partial_y\}$.

System of Imprimitivity $y = \alpha x + const.$

$\mathbf{g}_{27} : \{\partial_y\}$.

3.3 The Primitive Space Groups

Lie gave a partial classification of the point groups of the three-dimensional space. Among them are all primitive space groups. The listing given below is taken from Chapter 7 in [8] ("Bestimmung aller primitiven Gruppen des dreifach ausgedehnten Raumes"). Let $G := \partial_x, \partial_y, \partial_z$.

$\mathbf{p}_1 : \{G, (x\partial_v, y\partial_v, z\partial_v)_{v=x,y,z}, (u^2\partial_u + uv\partial_v + uw\partial_w)_{\{u,v,w\}=\{x,y,z\}}\}$, size: 15.

$\mathbf{p}_2 : \{G, (x\partial_v, y\partial_v, z\partial_v)_{v=x,y,z}\}$, size: 12.

$\mathbf{p}_3 : \{G, (x\partial_x - v\partial_v)_{v=y,z}, (v\partial_u, w\partial_u)_{\{u,v,w\}=\{x,y,z\}}\}$, size: 11.

$\mathbf{p}_4 : \{\partial_z, x\partial_y, y\partial_x, x\partial_x - y\partial_y, x\partial_x + y\partial_y + 2z\partial_z, z(x\partial_x + y\partial_y + z\partial_z), g, g_z, \bar{g}, \bar{g}_z\}$,
$g := z\partial_x - y(x\partial_x + y\partial_y + z\partial_z)$, $\bar{g} := z\partial_y + x(x\partial_x + y\partial_y + z\partial_z)$, size: 10.

$\mathbf{p}_5 : \{G, (u\partial_v - v\partial_u)_{(u,v)=(x,y),(x,z),(y,z)}\}$, size: 6.

$\mathbf{p}_6 : \{x\partial_x + y\partial_y + z\partial_z\} \cup \mathbf{p}_5$, size: 7.

$\mathbf{p}_7 : \{(\partial_u + v\partial_z)_{\{u,v\}=\{x,y\}}, (v\partial_v + z\partial_z)_{v=x,y}, (u^2\partial_u + (xy - z)\partial_v + uz\partial_z)_{\{u,v\}=\{x,y\}}\}$,
size: 6.

$\mathbf{p}_8 : \{G, (u\partial_v - v\partial_u)_{(u,v)=(x,y),(x,z),(y,z)}, g, (2ug - S(x,y,z)\partial_v)_{(u,v)=(x,y),(x,z),(y,z)}\}$,
$g := x\partial_x + y\partial_y + z\partial_z$, $S(x,y,z) := x^2 + y^2 + z^2$, size: 10.

3.4 Lie's Imprimitive Space Groups

In his partial classification of the point groups of the three-dimensional space, Lie divided the imprimitive space groups into three categories, according to their systems of imprimitivity. He computed the groups of the first two categories explicitly; we present them in this section. They are extracted from Chapter 8 in [8] ("Bestimmung gewisser imprimitiver Gruppen des dreifach ausgedehnten Raumes").

System of Imprimitivity $\varphi(x, y, z) = const.$

$\mathbf{ip}_1 : \{\partial_x, \partial_y, (u\partial_v)_{(u,v)\in\{x,y\}^2}, (u^2\partial_u + xy\partial_v)_{\{u,v\}=\{x,y\}}\}$, size: 8.

$\mathbf{ip}_2 : \{y\partial_x, x\partial_y, x\partial_x - y\partial_y, (Z_i(z)\partial_v)_{i\to l}^{v=x,y}\}$, $l \geq 1$, $Z_1 = 1$, size: $2l + 3$.

$\mathbf{ip}_3 : \{x\partial_x + y\partial_y\} \cup \mathbf{ip}_2$, size: $2l + 3$.

$\mathbf{ip}_4 : \{\partial_z\} \cup \mathbf{ip}_1$, size: 9.

$\mathbf{ip}_5 : \{\partial_z, x\partial_y, y\partial_x, x\partial_x - y\partial_y, (z^i e^{\lambda_k z}\partial_v)_{k\to h,\ i\to^* m_k}^{v=x,y}\}$, $h \geq 1$, $m_i \geq 0$,

λ_k pairwise disjoint integers, w.l.o.g. $\lambda_1 = 0$, size: $4 + 2h + 2\sum_{k=1}^{h} m_k$.

$\mathbf{ip}_6 : \{x\partial_x + y\partial_y\} \cup \mathbf{ip}_5$, size: $5 + 2h + 2\sum_{k=1}^{h} m_k$.

$\mathbf{ip}_7 : \{z\partial_z\} \cup \mathbf{ip}_4$, size: 10.

$\mathbf{ip}_8 : \{\partial_z, x\partial_y, y\partial_x, x\partial_x - y\partial_y, z\partial_z + a(x\partial_x + y\partial_y), (z^i\partial_v)_{i\to^* m}^{v=x,y}\}$, size: $2m + 7$.

$\mathbf{ip}_9 : \{x\partial_y, y\partial_x, x\partial_x - y\partial_y, x\partial_x + y\partial_y, (z^i\partial_z)_{i=0}^{1}, (z^i\partial_v)_{i\to^* m}^{v=x,y}\}$, size: $2m + 8$.

$\mathbf{ip}_{10} : \{z^2\partial_z\} \cup \mathbf{ip}_7$, size: 11.

$\mathbf{ip}_{11} : \{\partial_z, x\partial_y, y\partial_x, x\partial_x - y\partial_y, g, g_z, (z^i\partial_v)_{i\to^* m}^{v=x,y}\}$,
$g := z^2\partial_z + mz(x\partial_x + y\partial_y)$, size: $2m + 8$.

$\mathbf{ip}_{12} : \{z^2\partial_z + mz(x\partial_x + y\partial_y)\} \cup \mathbf{ip}_9$, size: $2m + 9$.

Two System of Imprimitivity $\varphi(x, y, z) = const.$, $\psi(x, y, z) = const.$

$\mathbf{ip}_{13} : \{\partial_x, \partial_y, x\partial_y + \partial_z, x\partial_x - y\partial_y - 2z\partial_z, y\partial_x - z^2\partial_z\}$.

$\mathbf{ip}_{14} : \{\partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y\} = (\mathbf{ip}_2)_{l=1}$.

$\mathbf{ip}_{15} : \{\partial_z\} \cup \mathbf{ip}_{14} = (\mathbf{ip}_5)_{h=1}^{m_1=\lambda_1=0}$, size: 6.

$\mathbf{ip}_{16} : \{\partial_z, \partial_x, (x^{i-1}\partial_y + \frac{1}{i}x^i\partial_z)_{i=1}^{2}, x\partial_x - y\partial_y, y\partial_x + \frac{1}{2}y^2\partial_z\}$.

$\mathbf{ip}_{17} : \{(x^\rho y^\sigma \partial_z)_{\sigma+\rho\to^* h}\} \cup \mathbf{ip}_{14}$, $(\mathbf{ip}_{17})_{h=0} = \mathbf{ip}_{15}$, size: $h + 6$.

$\mathbf{ip}_{18} : \{z\partial_z\} \cup \mathbf{ip}_{17}$, $(\mathbf{ip}_{18})_{h=0} = (\mathbf{ip}_8)_{a=m=0}$, size: $h + 7$.

$\mathbf{ip}_{19} : \{(z^i\partial_z)_{i=0}^{2}\} \cup \mathbf{ip}_{14} = (\mathbf{ip}_{11})_{m=0}$, size: 8.

$\mathbf{ip}_{20} : \{x\partial_x + y\partial_y\} \cup \mathbf{ip}_{13}$, size: 6.

$\mathbf{ip}_{21} : \{x\partial_x + y\partial_y\} \cup \mathbf{ip}_{14} = (\mathbf{ip}_3)_{l=1}$, size: 6.

$\mathbf{ip}_{22} : \{x\partial_x + y\partial_y + \partial_z\} \cup \mathbf{ip}_{14}$, size: 6.

$\mathbf{ip}_{23} : \{x\partial_x + y\partial_y + az\partial_z\} \cup \mathbf{ip}_{17}$, size: $h + 7$,
$(\mathbf{ip}_{23})_{a=h=0} = (\mathbf{ip}_6)_{h=1}^{m_1=\lambda_1=0}$, $(\mathbf{ip}_{23})_{a\neq 0}^{h=0} = (\mathbf{ip}_8)_{m=0}$.

$\mathbf{ip}_{24}:\{x\partial_x+y\partial_y+2z\partial_z\}\cup\mathbf{ip}_{16}$, size: 7.

$\mathbf{ip}_{25}:\{x\partial_x+y\partial_y\}\cup\mathbf{ip}_{18}$, $(\mathbf{ip}_{25})_{h=0}=\mathbf{ip}_{9m=0}$, size: $h+8$.

$\mathbf{ip}_{26}:\{x\partial_x+y\partial_y\}\cup\mathbf{ip}_{19}=(\mathbf{ip}_{12})_{m=0}$, size: 9.

$\mathbf{ip}_{27}:\{(u^2\partial_u+xy\partial_v+z^{[u=y]}(y-xz)\partial_z)_{\{u,v\}=\{x,y\}}\}\cup\mathbf{ip}_{20}$, size: 8.

$\mathbf{ip}_{28}:\{(u^2\partial_u+xy\partial_v)_{\{u,v\}=\{x,y\}}\}\cup\mathbf{ip}_{21}$, size: 8.

$\mathbf{ip}_{29}:\{(u^2\partial_u+xy\partial_v+\frac{3}{2}u\partial_z)_{\{u,v\}=\{x,y\}}\}\cup\mathbf{ip}_{22}=\mathbf{ip}_1$, size: 8.

$\mathbf{ip}_{30}:\{(u^2\partial_u+xy\partial_v+huz\partial_z)_{\{u,v\}=\{x,y\}}\}\cup(\mathbf{ip}_{23})_{a=2h/3}$,
$(\mathbf{ip}_{30})_{h=0}=\mathbf{ip}_4$, size: $h+9$.

$\mathbf{ip}_{31}:\{z\partial_z,(u^2\partial_u+xy\partial_v+uz\partial_z)_{\{u,v\}=\{x,y\}}\}\cup\mathbf{ip}_{21}$, size: 9.

$\mathbf{ip}_{32}:\{(u^2\partial_u+xy\partial_v+huz\partial_z)_{\{u,v\}=\{x,y\}}\}\cup\mathbf{ip}_{25}$, $(\mathbf{ip}_{32})_{h=0}=\mathbf{ip}_7$, size: $h+10$.

$\mathbf{ip}_{33}:\{(u^2\partial_u+xy\partial_v)_{\{u,v\}=\{x,y\}}\}\cup\mathbf{ip}_{26}=\mathbf{ip}_{10}$, size: 11.

3.5 Amaldis Imprimitive Space Groups

Lie gave in his book [8] not all imprimitive space groups explicitly. He did so for the groups of categories I and II that are listed in the previous Section 3.4. He only gave two methods on how to compute the *groups of category III*. They have two systems of imprimitivity of the form

$$\varphi(x,y,z)=const.,\psi(x,y,z)=const. \tag{3.1}$$

In addition, these systems can be written as a system of surfaces of the form

$$\Omega(\varphi(x,y,z),\psi(x,y,z))=const. \tag{3.2}$$

This category of groups is actually much larger than the first two categories. It was the Italian mathematician Ugo Amaldi who explicitly computed the representatives by the first method proposed by Lie [1]. To this end, let us consider an m-parameter space group G_m, which is doubly imprimitive in the above sense. First we apply a point transformation to G_m that transforms the invariant congruence (3.1) into the star

$$x = const., \ y = const., \tag{3.3}$$

the set of lines parallel to the z-axis. Furthermore, it should transform the system of surfaces (3.2) into

$$x = const.,$$

the bundle of planes normal to the x-axis. This amounts to saying that the generators X_i of G_m now have the following form:

$$X_i = \xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z, \qquad i = 1, \dots, m.$$

Their projections

$$X_i^{(p)} = \xi_i(x)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1, \dots, m,$$

which are not necessarily linear independent, form an imprimitive group in two variables of dimension $l \le m$. Since the groups of the plane are already classified (compare Section 3.2), we assume w.l.o.g. that the projected group of G_m represents one of those known forms. Hence the generators X_i of G_m have the following form:

$$\begin{aligned} X_i &= \xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z, \qquad i = 1, \dots, l, \\ X_i &= \qquad\qquad\qquad\qquad\qquad \varphi_i(x,y,z)\partial_z, \qquad j = 1, \dots, m-l. \end{aligned}$$

The $\varphi_i(x,y,z)\partial_z$ generate a subgroup of G_m, the biggest subgroup that leaves all lines of the form (3.3) invariant. Each single line allows one of four cases, namely the identity transformation, or a group with one, two, or three parameters (compare [8], page 6). As a consequence, the subgroup generated by the $\varphi_i(x,y,z)\partial_z$ is equivalent to the identity transformation, or to a subgroup with generators of one of the following three forms (compare [8], page 155):

$$\begin{aligned} \varphi_j(x,y)\partial_z, &\qquad j = 1, 2, \dots, m-l, \text{ or} \\ \partial_z, \varphi_j(x,y)\partial_z, &\qquad j = 1, 2, \dots, m-l-1, \text{ or} \\ z^j\partial_z, &\qquad j = 0, 1, 2. \end{aligned}$$

Corresponding to those four cases, we get the following possible forms for G_m:

$$\begin{aligned}
\xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z \qquad & (i = 1, 2, \dots, l). && ([A]) \\
\xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z \qquad & (i = 1, 2, \dots, l), && ([B]) \\
\varphi_j(x,y)\partial_z \qquad & (j = 1, 2, \dots, h > 0). && \\
\xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z \qquad & (i = 1, 2, \dots, l), && ([C]) \\
z\partial_z,\ \varphi_j(x,y)\partial_z \qquad & (j = 1, 2, \dots, h > 0). && \\
\xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z \qquad & (i = 1, 2, \dots, l), && ([D]) \\
z^j\partial_z \qquad & (j = 0, 1, 2). &&
\end{aligned}$$

Conclusion Starting with the imprimitive groups of the plane, one has to consider each of the cases [A], [B], [C] and [D] in order to obtain the corresponding groups of the space. In the cases [A] and [D], this reduces to determining the functions $\zeta_i(x,y,z)$ such that all commutators

$$X_iX_j - X_jX_i$$

belong to the linear hull of the respective generators.

Amaldi also performed further precalculations revealing the structure of the unknown functions more closely.

Category B: For a group of type [B], we can choose

$$\zeta_i(x,y,z) = \zeta_{i,1}(x,y)z + \zeta_{i,2}(x,y), \qquad i = 1, 2, \dots, l.$$

Category C: For a group of type [C], we can choose

$$\zeta_i(x,y,z) = \bar{\zeta}_i(x,y)z, \qquad i = 1, 2, \dots, l.$$

Category D: For a group of type [D], we can choose

$$\zeta_i(x,y,z) = 0, \qquad i = 1, 2, \dots, l.$$

Concerning the functions $\varphi_j(x, y)$ that occur in types [B] and [C], Amaldi derived some theorems on differential systems with constant coefficients for later use, compare pages 279-282 in [1].

Theorem If an h-dimensional linear real function space S_h of functions of x and y is transformed into itself by the operations ∂_x, ∂_y, it is possible to choose a fundamental basis in S_h, made up of a finite number of groups of functions of the form

$$e^{ax+by}x^m y^n, \qquad n = 0, \ldots, r \quad m = 0, \ldots, h_1 + \ldots + h_{r-n+1}.$$

Two groups are distinguished by at least one of the constants a, b, and in general also by the maximal values of the exponents m, n.

Corollary If an h-dimensional linear real function space S_h of functions of x and y is transformed into itself by the operations $x\partial_x$, $y\partial_y$, it is possible to choose a fundamental basis in S_h, made up of a finite number of groups of functions of the form

$$x^a y^b \log x^m \log y^n, \qquad n = 0, \ldots, r, \quad m = 0, \ldots, h_1 + \ldots + h_{r-n+1}.$$

Two groups are distinguished by at least one of the constants a, b, which are roots of the respective fundamental equations of $x\partial_x$, $y\partial_y$ in S_h, and in general also by the maximal values of the exponents m, n.

3.5.1 Amaldis Imprimitive Plane Groups

Amaldis listing of the imprimitive groups of the plane does not strictly follow Lie's classification given in Section 3.2. Not considering the structure of invariant fibers of curves, any parameter is given free variability. By this, the number of groups is reduced to the minimum. In addition, the order in which the groups are listed has been chosen to shorten the calculations, which are inevitably long and hard anyway. Where it was possible, any group G_l is followed by the minimal group G_{l+t} in which it is contained. This has the following advantage: suppose the corresponding space groups for G_l are already known. If

$$X_1, \ldots, X_t$$

are the generators that must be added to G_l to obtain G_{l+t}, then by adding

$$X_i + \zeta_i(x, y, z)\partial_z, \qquad i = 1, 2, \ldots, t$$

to the corresponding and already calculated space groups of G_l, we reduce the calculations for G_{l+t} to the calculations of the remaining functions ζ_i. Following these considerations, the following listing of groups is obtained.

$\mathbf{ip}_1 : \{(x^i\partial_y)_{i=0}^s, \partial_x, x\partial_x + cy\partial_y\}$, $s \geq 1$, size: $s + 3$.

$\mathbf{ip}_2 : \{(x^i\partial_y)_{i=0}^s, \partial_x, g, g_x\}$, $s \geq 1$, $g := x^2\partial_x + sxy\partial_y$, size: $s + 4$.

$\mathbf{ip}_3 : \{(x^i\partial_y)_{i=0}^{s-1}, \partial_x, x\partial_x + (sy + x^s)\partial_y\}$, $s \geq 1$, size: $s + 2$.

$\mathbf{ip}_4 : \{(x^i\partial_y)_{i=0}^s, y\partial_y, \partial_x, x\partial_x\}$, $s \geq 1$, size: $s + 4$.

$\mathbf{ip}_5 : \{(x^i\partial_y)_{i=0}^s, y\partial_y, (x^i\partial_x + [i = 2]sxy\partial_y)_{i=0}^2\}$, $s \geq 1$, size: $s + 5$.

$\mathbf{ip}_6 : \{y\partial_y, (x^i\partial_x + [i = 2]xy\partial_y)_{i=0}^2\}$.

$\mathbf{ip}_7 : \{\partial_x, g, g_x\}$, $g := x^2\partial_x + xy\partial_y$.

$\mathbf{ip}_8 : \{(x^i\partial_x + y^i\partial_y)_{i=0}^2\}$.

$\mathbf{ip}_9 : \{\partial_y\}$.

$\mathbf{ip}_{10} : \{\partial_y, y\partial_y\}$.

$\mathbf{ip}_{11} : \{\partial_x, \partial_y\}$.

$\mathbf{ip}_{12} : \{\partial_y, x\partial_x + y\partial_y\}$.

$\mathbf{ip}_{13} : \{\partial_y, y\partial_y, \partial_x\}$.

$\mathbf{ip}_{14} : \{(y^i\partial_y)_{i=0}^2\}$.

$\mathbf{ip}_{15} : \{(y^i\partial_y)_{i=0}^2, \partial_x\}$.

$\mathbf{ip}_{16} : \{(y^i\partial_y)_{i=0}^2, \partial_x, x\partial_x\}$.

$\mathbf{ip}_{17} : \{(y^i\partial_y)_{i=0}^2, (x^i\partial_x)_{i=0}^2\}$.

$\mathbf{ip}_{18} : \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^s\}$, $s \geq 0$, size: $s + 2$.

$\mathbf{ip}_{19} : \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^s, y\partial_y\}$, $s \geq 0$, size: $s + 3$.

$\mathbf{ip}_{20} : \{(x^j e^{a_i x}\partial_y)_{i \to l}^{j \to^* s_i}, \partial_x\}$, $a_1 \in \{0, 1\}$, $l + \sum s_i \geq 2$, size: $1 + l + \sum_{i=1}^l s_i$.

$\mathbf{ip}_{21} : \{(x^j e^{a_i x}\partial_y)_{i \to^* l,\ j \to^* s_i}^{a_0 = 0,\ a_1 = 1}, y\partial_y, \partial_x\}$, $l \geq 0$, $l + \sum s_i \geq 0$, size: $2 + l + \sum_{i=1}^l s_i$.

3.5.2 Example Calculations

We demonstrate the calculation of the space groups of type [A] corresponding to $\mathbf{ip}_1$, denoted by $\mathbf{ip}_{1,A}$. This space group will be of the form

$$\{\partial_y + \alpha_0\partial_z, x\partial_y + \alpha_1\partial_z, \dots, x^s\partial_y + \alpha_s\partial_z, \partial_x + \beta_0\partial_z, x\partial_x + cy\partial_y + \beta_1\partial_z\},$$

where $\alpha_i = \alpha_i(x, y, z)$, $\beta_j = \beta_j(x, y, z)$ for $i = 0, \dots, s$ and $j = 0, 1$. Considering the commutator

$$[\partial_x + \beta_0\partial_z, \partial_y + \alpha_0\partial_z] = (\frac{\partial\alpha_0}{\partial x} + \beta_0\frac{\partial\alpha_0}{\partial z} - \frac{\partial\beta_0}{\partial y} - \alpha_0\frac{\partial\beta_0}{\partial z})\partial_z,$$

we conclude that

$$\frac{\partial\alpha_0}{\partial x} + \beta_0\frac{\partial\alpha_0}{\partial z} - \frac{\partial\beta_0}{\partial y} - \alpha_0\frac{\partial\beta_0}{\partial z} = 0.$$

This is the condition such that the two linear equations

$$\frac{\partial\zeta}{\partial y} + \alpha_0\frac{\partial\zeta}{\partial z} = 0, \quad \frac{\partial\zeta}{\partial x} + \beta_0\frac{\partial\zeta}{\partial z} = 0$$

form a complete system; the common solution ζ is guaranteed to depend on z (compare [7], page 91ff). Choosing this function ζ as new z, the generators $\partial_x + \beta_0\partial_z$, $\partial_y + \alpha_0\partial_z$ are reduced to ∂_x, ∂_y, respectively, while the remaining generators do not change their nature. So we assume $\mathbf{ip}_{1,A}$ to be of the form

$$\{\partial_y, x\partial_y + \alpha_1\partial_z, \dots, x^s\partial_y + \alpha_s\partial_z, \partial_x, x\partial_x + cy\partial_y + \beta_1\partial_z\}.$$

By considering the commutators

$$[\partial_x,\ x\partial_x + cy\partial_y + \beta_1\partial_z] = \partial_x + \frac{\partial\beta_1}{\partial x}\partial_z,$$
$$[\partial_y, x\partial_x + cy\partial_y + \beta_1\partial_z] = c\partial_y + \frac{\partial\beta_1}{\partial y}\partial_z$$

we conclude that $\beta_1 = \beta_1(z)$. From

$$[\partial_x,\ x^i\partial_y + \alpha_i\partial_z] = ix^{i-1}\partial_y + \frac{\partial\alpha_i}{\partial x}\partial_z, \quad [\partial_y, x^i\partial_y + \alpha_i\partial_z] = \frac{\partial\alpha_i}{\partial y}\partial_z$$

we conclude that

$$\alpha_i = \alpha_i(x, z), \quad \frac{\partial\alpha_i}{\partial x} = i\alpha_{i-1}, \qquad i = 1, \dots, s.$$

Let now $t \leq s$ be such that $\alpha_0 = \alpha_1 = \ldots = \alpha_{t-1} = 0$, $\alpha_t \neq 0$. Then $\alpha_t = \alpha_t(z)$ and we can reduce α_t to 1 by choosing $\int \frac{dz}{\alpha_t}$ as new z. From

$$[x^t\partial_y + \partial_z,\ x\partial_x + cy\partial_y + \beta_1\partial_z] = (c-t)x^t\partial_y + \frac{\partial\beta_1}{\partial z}\partial_z$$

we have that $\beta_1 = (c-t)z + b_1$.

Case $c \neq t$: we choose $z + \frac{b_1}{c-t}$ as new z and thereby reduce b_1 to 0. As above, by considering

$$[x^i\partial_y + \alpha_i\partial_z,\ x\partial_x + cy\partial_y + (c-t)z\partial_z]$$

we conclude that $x\frac{d\alpha_i}{dx} = (i-t)\alpha_i$. Together with $\frac{d\alpha_i}{dx} = i\alpha_{i-1}$ this implies $ix\alpha_{i-1} = (i-t)\alpha_i$. Hence we get

$\mathbf{ip}^1_{1,A} : \{\partial_y, x\partial_y, \ldots, x^{t-1}\partial_y, x^t\partial_y + \binom{t}{t}x^0\partial_z, x^{t+1}\partial_y + \binom{t+1}{t}x^1\partial_z, \ldots,$
$x^s\partial_y + \binom{s}{t}x^{s-t}\partial_z, \partial_x, x\partial_x + cy\partial_y + (c-t)z\partial_z\}$, $1 \leq t \leq s$.

Case $c = t$: in this case we have $\beta_1 = const.$, and hence

$\mathbf{ip}^2_{1,A} : \{\partial_y, x\partial_y, \ldots, x^{t-1}\partial_y, x^t\partial_y + \binom{t}{t}x^0\partial_z, x^{t+1}\partial_y + \binom{t+1}{t}x^1\partial_z, \ldots,$
$x^s\partial_y + \binom{s}{t}x^{s-t}\partial_z, \partial_x, x\partial_x + ty\partial_y + c\partial_z\}$, $1 \leq t \leq s$.

As an additional example we consider the calculation of $\mathbf{ip}_{5,A}$, a space group where Amaldis calculation was slightly erroneous. First of all we note that $\mathbf{ip}_4$ and $\mathbf{ip}_5$ differ only by the additional generator $x^2\partial_x + sxy\partial_y$:

$$\mathbf{ip}_4[s] = \{\partial_y, x\partial_y, \ldots, x^s\partial_y, y\partial_y, \partial_x, x\partial_x\},$$
$$\mathbf{ip}_5[s] = \{\partial_y, x\partial_y, \ldots, x^s\partial_y, y\partial_y, \partial_x, x\partial_x, x^2\partial_x + sxy\partial_y\}.$$

Furthermore, we suppose to already know the corresponding space group to $\mathbf{ip}_4$ of type [A]:

$$\mathbf{ip}_{4,A}[s,t] = \{\partial_y, x\partial_y, \ldots, x^{t-1}\partial_y, x^t\partial_y + \binom{t}{t}x^0\partial_z, x^{t+1}\partial_y + \binom{t+1}{t}x^1\partial_z, \ldots, x^s\partial_y + \binom{s}{t}x^{s-t}\partial_z, \partial_x, y\partial_y + z\partial_z, x\partial_x - tz\partial_z\}.$$

Following the comment in the previous subsection, our ansatz for $\mathbf{ip}_5$ is of the form

$$\mathbf{ip}_{5,A}[s,t] = \{\partial_y, x\partial_y, \dots, x^{t-1}\partial_y, x^t\partial_y + \binom{t}{t}x^0\partial_z, x^{t+1}\partial_y + \binom{t+1}{t}x^1\partial_z, \dots$$
$$x^s\partial_y + \binom{s}{t}x^{s-t}\partial_z, \partial_x, y\partial_y + z\partial_z, x\partial_x - tz\partial_z, x^2\partial_x + sxy\partial_y + \zeta(x,y,z)\partial_z\}.$$

By considering

$$[\partial_y,\ x^2\partial_x + sxy\partial_y + \zeta(x,y,z)\partial_z] = sx\partial_x + \zeta_y\partial_z$$

we conclude that the cases $t > 1$ are not possible (this point was overlooked by Amaldi) and that $\zeta_y = s$, i.e. $\zeta(x,y,z) = sy + \bar{\zeta}(x,z)$. By considering

$$[\partial_x,\ x^2\partial_x + sxy\partial_y + (sy + \bar{\zeta}(x,z))\partial_z] = 2x\partial_x + sy\partial_y + \bar{\zeta}_x\partial_z$$

we conclude that $\bar{\zeta}_x = (s-2)z$, i.e. $\bar{\zeta}(x,z) = (s-2)xz + \overline{\zeta}(z)$. Finally, by considering

$$[x\partial_y + \partial_z,\ x^2\partial_x + sxy\partial_y + (sy + (s-2)xz + \overline{\zeta}(z))\partial_z] =$$
$$(s-1)x^2\partial_y + 2(s-1)x\partial_z + \overline{\zeta}_z\partial_z$$

we conclude that $\overline{\zeta}_z = 0$, i.e. $\overline{\zeta}(z) = C$, w.l.o.g. $\overline{\zeta}(z) = 0$. Hence we arrive at

$$\mathbf{ip}_{5,A}[s] = \{\partial_x, \partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, \dots, x^s\partial_y + sx^{s-1}\partial_z,$$
$$x\partial_x - z\partial_z, y\partial_y + z\partial_z, x^2\partial_x + sxy\partial_y + (sy + (s-2)xz)\partial_z\}.$$

3.5.3 Amaldis Groups of Type A

We recall that the n-th Amaldi space group(s) of type A, denoted by $\mathbf{ip}_{n,A}$ with optional additional indices, is/are represented by generators of the form

$$\xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z, \qquad i = 1, 2, \dots, l,$$

where

$$\xi_i(x)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1, 2, \dots, l$$

are the generators of the n-th Amaldi plane group, as listed in Section 3.5.1 ("Amaldis Imprimitive Plane Groups").

$\mathbf{ip}^1_{1,A} : \{(x^i\partial_y + [i \geq t]\binom{i}{t}x^{i-t}\partial_z)^s_{i=0}, \partial_x, x\partial_x + cy\partial_y + (c-t)z\partial_z\}$,
$0 \leq t-1 \leq s$, size: $s+3$.

$\mathbf{ip}^2_{1,A} : \{(x^i\partial_y + [i \geq t]\binom{i}{t}x^{i-t}\partial_z)^s_{i=0}, \partial_x, x\partial_x + ty\partial_y + c\partial_z\}$,
$0 \leq t-1 \leq s$, size: $s+3$.

$\mathbf{ip}^1_{2,A} : \{(x^i\partial_y + ix^{i-1}\partial_z)^s_{i=0}, \partial_x, g, g_x\}$, $s \geq 0$,
$g := x^2\partial_x + sxy\partial_y + [(s-2)zx + sy]\partial_z$, size: $s+4$.

$\mathbf{ip}^2_{2,A} : \{(x^i\partial_y)^s_{i=0}, \partial_x, g, g_x\}$, $s \geq 1$,
$g := x^2\partial_x + sxy\partial_y + z(x + cz^2)\partial_z$, size: $s+4$.

$\mathbf{ip}^3_{2,A} : \{\partial_x, \partial_y, g, \frac{1}{2}g_x, \bar{g}, \frac{1}{2}\bar{g}_x\}$, $g := x^2\partial_y + 2x\partial_z$, $\bar{g} := x^2\partial_x + 2xy\partial_y + 2(cx+y)\partial_z$.

$\mathbf{ip}^1_{3,A} : \{(x^i\partial_y + [i \geq t]\binom{i}{t}x^{i-t}\partial_z)^{s-1}_{i=0}, \partial_x, x\partial_x + (sy + x^s)\partial_y + [(s-t)z + \binom{s}{t}x^{s-t}]\partial_z\}$,
$1 \leq t < s$, size: $s+2$.

$\mathbf{ip}^2_{3,A} : \{(x^i\partial_y)^{s-1}_{i=0}, \partial_x, x\partial_x + (sy + x^s)\partial_y + c\partial_z\}$, $s \geq 1$, size: $s+2$.

$\mathbf{ip}_{4,A} : \{(x^i\partial_y + [i \geq t]\binom{i}{t}x^{i-t}\partial_z)^s_{i=0}, \partial_x, y\partial_y + z\partial_z, x\partial_x - tz\partial_z\}$,
$0 \leq t-1 \leq s$, size: $s+4$.

$\mathbf{ip}_{5,A} : \{\partial_x, (x^i\partial_y + ix^{i-1}\partial_z)^s_{i=0}, x\partial_x - z\partial_z, y\partial_y + z\partial_z, g\}$, $s \geq 0$,
$g := x^2\partial_x + sxy\partial_y + (sy + (s-2)xz)\partial_z$, size: $s+5$.

$\mathbf{ip}_{6,A} : \{\partial_x, y\partial_y, g, \frac{1}{2}(g_x - y\partial_y)\}$, $g := x^2\partial_x + xy\partial_y - 2axz\partial_z$, $a \in \{0, 1\}$.

$\mathbf{ip}_{7,A} : \{\partial_x, g, g_x\}$, $g := x^2\partial_x + xy\partial_y + y^2\partial_z$.

$\mathbf{ip}_{8,A} : \{(x^i\partial_x + y^i\partial_y + [i=2]c(y-x)\partial_z)^2_{i=0}\}$.

$\mathbf{ip}_{9,A} : \{\partial_y\}$.

$\mathbf{ip}_{10,A} : \{\partial_x, y\partial_y + \partial_z\}$.

$\mathbf{ip}_{11,A} : \{\partial_x, \partial_y\}$.

$\mathbf{ip}_{12,A} : \{\partial_y, x\partial_x + y\partial_y + c\partial_z\}$.

$\mathbf{ip}_{13,A} : \{\partial_x, \partial_y, y\partial_y + \partial_z\}$.

$\mathbf{ip}^1_{14,A}: \{\partial_y, g, \frac{1}{2}g_y\},\ g := y^2\partial_y + (2zy + xz^2)\partial_z.$

$\mathbf{ip}^2_{14,A}: \{\partial_y, g, \frac{1}{2}g_y\},\ g := y^2\partial_y + (2zy + cz^2)\partial_z.$

$\mathbf{ip}_{15,A}: \{\partial_x, \partial_y, g, \frac{1}{2}g_y\},\ g := y^2\partial_y + (2y + cz)z\partial_z.$

$\mathbf{ip}^1_{16,A}: \{\partial_x, \partial_y, x\partial_x, g, \frac{1}{2}g_y\},\ g := y^2\partial_y + (2y + cz)z\partial_z.$

$\mathbf{ip}^2_{16,A}: \{\partial_x, \partial_y, x\partial_x + cz\partial_z, g, \frac{1}{2}g_y\},\ g := y^2\partial_y + 2zy\partial_z.$

$\mathbf{ip}^3_{16,A}: \{\partial_x, \partial_y, x\partial_x + z\partial_z, g, \frac{1}{2}g_y\},\ g := y^2\partial_y.$

$\mathbf{ip}^1_{17,A}: \{\partial_y, (x^i\partial_x)^2_{i=0}, g, \frac{1}{2}g_y\},\ g := y^2\partial_y + (2y + cz)z\partial_z.$

$\mathbf{ip}^2_{17,A}: \{\partial_x, \partial_y, g, \frac{1}{2}g_y, \bar{g}, \frac{1}{2}\bar{g}_x\}, g := y^2\partial_y + 2zy\partial_z,\ \bar{g} := x^2\partial_x + 2czx\partial_z.$

$\mathbf{ip}_{18,A}: \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y + [i \geq t]\varphi_i(x)\partial_z)^s_{i=1}\},$
$\varphi_t = 1,\ 0 \leq t - 1 \leq s$, size: $s + 2$.

$\mathbf{ip}_{19,A}: \{y\partial_y + z\partial_z\} \cup \mathbf{ip}_{18,A}$, size: $s + 3$.

$\mathbf{ip}_{20,A}: \{\partial_x, (e^{a_i x}(x^j\partial_y + \pi_{i,j}\partial_z))^{j\to^* s_i}_{i\to q}\},\ \pi_{i,j} = \sum_{k=0}^{j} \gamma_{i,k}\binom{j}{k}x^{j-k},$
w.l.o.g. $(\gamma_{1,0}, \gamma_{1,1}) = (0, 1)$, size: $1 + q + \sum_{i=1}^{q} s_i > 2$.

$\mathbf{ip}_{21,A}: \{y\partial_y + z\partial_z, (x^i\partial_y + \pi_{0,i}\partial_z)^s_{i=0}\} \cup \mathbf{ip}_{20,A}[(s_1, .., s_h), a_1 = 1],$
size: $3 + h + s + \sum_{i=1}^{h} s_i \geq 3$.

3.5.4 Amaldis Groups of Type B

We repeat that the n-th Amaldi space group(s) of type B, denoted by $\mathbf{ip}_{n,B}$ with optional additional indices, is/are represented by generators of the form

$$\begin{aligned} \xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z, &\qquad i = 1, 2, \dots, l, \\ \varphi_j(x,y)\partial_z, &\qquad j = 1, 2, \dots, h > 0, \end{aligned}$$

where

$$\xi_i(x)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1, 2, \dots, l$$

are the generators of the n-th Amaldi plane group, as listed in Section 3.5.1 ("Amaldis Imprimitive Plane Groups"). Furthermore, $\zeta_i(x,y,z)$ can be chosen as

$$\zeta_i(x,y,z) = \zeta_{i,1}(x,y)z + \zeta_{i,2}(x,y), \qquad i = 1,2,\dots,l.$$

The Amaldi groups of type B have not been processed within the frame of this work. For matter of completeness, the have been extracted from [1] and listed here anyway. Features like the group size are not provided. For any further details, we refer to [1].

For integers k,l,m with $k \geq m$, we define $\langle k,l,m\rangle := \frac{k!(m+1)!}{l!(m+1+k-l)!}$

$\mathbf{ip}^1_{1,B}: \{\partial_x, (x^k\partial_y + [k \geq t]a\,\langle k,t,m_i\rangle\, x^{m_i+k-t+1}y^{n-i}\partial_z)^s_{k=0}, (x^m y^{n-j}\partial_z)^{m\leq m_j}_{j\to^* n},$
$x\partial_x + cy\partial_y + C(z)\partial_z\},$
$1 \leq t \leq s+1$, $m_{j+1} \geq m_j + s$ for $j \neq i$, $m_{i+1} \geq m_i + 2s - t$,
$C(z) := (m_i + 1 + c(n-i+1) - t)z.$

$\mathbf{ip}^2_{1,B}: \{\partial_x, (x^k\partial_y + [k \geq t]a\,\langle k,t,m_n\rangle\, x^{m_n+k-t+1}\partial_z)^s_{k=0}, (x^m y^{n-j}\partial_z)^{m\leq m_j}_{j\to^* n},$
$x\partial_x + cy\partial_y + C(z)\partial_z\},$
$C(z) := (m_n + c - t + 1)z$; $m_{j+1} \geq m_j + s.$

$\mathbf{ip}^3_{1,B}: \{\partial_x, (x^i\partial_y)^s_{i=0}, x\partial_x + cy\partial_y + c_1 z\partial_z, (x^m y^{n-j}\partial_z)^{m\leq m_j}_{j\to^* n}\},$
$m_{j+1} \geq m_j + s$; $c_1 \in \{0,1\}.$

$\mathbf{ip}^1_{2,B}: \{\partial_z, \partial_x, \partial_y, x\partial_y, x^2\partial_y + cx\partial_z, x\partial_x + y\partial_y, x^2\partial_x + 2xy\partial_y + (cy + c_1 x)\partial_z\},$
c or $c_1 \in \{0,1\}.$

$\mathbf{ip}^2_{2,B}: \{\partial_x, (x^i\partial_y)^s_{i=0}, g, g_x, (x^i\partial_z)^s_{i=0}\}$, $g := x^2\partial_x + sxy\partial_y + sxz\partial_z.$

$\mathbf{ip}^3_{2,B}: \{\partial_z, (x^i\partial_y)^s_{i=0}, 2x\partial_x + sy\partial_y, x^2\partial_x + sxy\partial_y + cx\partial_z\}$, $c \in \{0,1\}$, $s \neq 2.$

$\mathbf{ip}^4_{2,B}: \{\partial_x, (x^i\partial_y)^s_{i=0}, g, g_x, (x^m y^{n-j})^{m\leq m_0+sj.}_{j\to^* n}\},$
$g := x^2\partial_x + sxy\partial_y + (m_0 + sn)xz\partial_z.$

$\mathbf{ip}^1_{3,B}: \{\partial_x, (x^k\partial_y + [k \geq t]a\,\langle k,t,m_i\rangle\, x^{m_i+k-t+1}y^{n-i}\partial_z)^{s-1}_{k=0}, (x^m y^{n-j}\partial_z)^{m\leq m_j}_{j\to^* n},$
$x\partial_x + (sy + x^s)\partial_y + C(x,y,z)\partial_z\},$

$m_{j+1} \geq m_j + s$ for $j \neq i$; $m_{i+1} \geq m_i + 2s - t$, $1 \leq t \leq s$,
$C(x,y,z) := [m_i + 1 + s(n-i+1) - t]z + \langle s, t, m_i \rangle \, x^{m_i+s-t+1} y^{n-i}$.

$\mathbf{ip}^2_{3,B}$: $\{\partial_x, (x^i \partial_y)_{i=0}^{s-1}, x\partial_x + (sy + x^s)\partial_y + cz\partial_z, (x^m y^{n-j} \partial_z)_{j \to^* n}^{m \leq m_j}\}$,
$m_{j+1} \geq m_j + s$; $c \in \{0, 1\}$.

$\mathbf{ip}^1_{4,B}$: $(x^k \partial_y + [k \geq t] a \binom{k}{t} x^{m_i+k-t+1} y^{n-i} \partial_z)_{k=0}^{s}, y\partial_y + C(x,z)\partial_z, \partial_x,$
$x\partial_x + \bar{C}(x,z)\partial_z, (x^m y^{n-j} \partial_z)_{j \to^* n}^{m \leq m_j}\}$,
$m_{j+1} \geq m_j + s$ for $j \neq i$, $m_{i+1} \geq m_i + 2s - t$,
$C(x,z) := (n-i+1)(z + cx^{m_n+1})$,
$\bar{C}(x,z) := (m_i + 1 - t)z + c(m_i - m_n - t)x^{m_n+1}$.

$\mathbf{ip}^2_{4,B}$: $\{(x^i \partial_y)_{i=0}^{s-1}, x^s \partial_y + C_1(x,y)\partial_z, x\partial_x + C_2(x,y,z)\partial_z, \partial_x, y\partial_y + C_3(x,z)\partial_z,$
$(x^m y^{n-j} \partial_z)_{j \to^* n}^{m \leq m_j}\}$, $m_{j+1} \geq m_j + s$,
$C_1(x,y) := a(n-i+1)x^{m_i+1} y^{n-i}$,
$C_2(x,y,z) := bz + c(b_i - m_n - 1)x^{m_n+1} + a(m_i + 1 - s - b)x^{m_i-s+1} y^{n-i+1}$,
$C_3(x,z) := (n-i+1)(z + cx^{m_n+1})$.

$\mathbf{ip}^3_{4,B}$: $\{(x^i \partial_y)_{i=0}^{s}, x\partial_x + C(x,y)\partial_z, \partial_x, y\partial_y + \bar{C}(x,y,z)\partial_z, (x^m y^{n-j} \partial_z)_{j \to^* n}^{m \leq m_j}\}$,
$m_0 \geq s$, $m_{j+1} > m_j + s$,
$C(x,y) := bz + \sum_{j=0}^{n} a_j (m_j + 1 - b) x^{m_j+1} y^{n-j} + aby^{n+1}$,
$\bar{C}(x,y,z) := cz + \sum_{j=0}^{n} a_j (n - j - c) x^{m_j+1} y^{n-j} + a(c - n - 1)y^{n+1}$.

$\mathbf{ip}^1_{5,B}$: $\{(x^i \partial_y)_{i=0}^{s-1}, x^s \partial_y + ax^{s-1} y^n \partial_z, x\partial_x - C_1(x,z)\partial_z, \partial_x, y\partial_y + C_2(x,z)\partial_z,$
$x^2 \partial_x + sxy\partial_y + C_3(x,z)\partial_z, (x^m y^{n-j} \partial_z)_{j \to^* n}^{m \leq (j+1)s-2}\}$, $s \geq 2$,
$C_1(x,z) := z + c(n+1)sx^{s(n+1)-1}$,
$C_2(x,z) := (n+1)(z + cx^{s(n+1)-1})$,
$C_3(x,y,z) := (s(n+1) - 2)xz + \frac{a}{n+1} y^{n+1} - cx^{s(n+1)}$.

$\mathbf{ip}^2_{5,B}$: $\{(x^i \partial_y)_{i=0}^{s}, x\partial_x + C_1(x,y,z)\partial_z, y\partial_y + C_2(x,y,z)\partial_z, \partial_x,$
$x^2 \partial_x + sxy\partial_y + C_3(x,y,z)\partial_z, (x^m y^{n-j} \partial_z)_{j \to^* n}^{m \leq m_0 + sj}\}$, $E_j = m_0 + sj$,
$C_1(x,y,z) := bz + \sum_{j=0}^{n} a_j (E_j - b) x^{E_j} y^{n-j} + aby^{n+1}$,

$C_2(x,y,z) := \frac{1}{s}\{(E_n - 2b)z - \sum_{j=0}^{n} a_j(E_j - 2b)x^{m_0+sj+1}y^{n-j}+$
$+a(E_{-1} - 2b)y^{n+1}\},$
$C_3(x,y,z) := E_n xz + \sum_{j=0}^{n} a_j x^{E_j+2}y^{n-j} + aE_{-1}y^{n-1}x.$

$\mathbf{ip}_{5,B}^{3} : \{(x^i\partial_y)_{i=0}^{s}, x\partial_x - C_1(x,y,z)\partial_z, y\partial_y + C_2(x,y,z)\partial_z, \partial_x,$
$x^2\partial_x + sxy\partial_y + C_3(x,y,z)\partial_z,(x^m y^{n-j}\partial_z)_{j\to^* n}^{m\leq(j+1)s-2}\},$
$C_1(x,y,z) := z - \sum_{j=0}^{n} s(j+1)a_j x^{s(j+1)-1}y^{n-j} + ay^{n+1},$
$C_2(x,y,z) := (n+1)z - \sum_{j=0}^{n} a_j(j+1)x^{s(j+1)-1}y^{n-j},$
$C_3(x,y,z) := (s(n+1)-2)xz + \sum_{j=0}^{n} a_j x^{s(j+1)}y^{n-j} - 2axy^{n+1} + b'y^{n+1}.$

$\mathbf{ip}_{5,B}^{4} : \{(x^i\partial_y)_{i=0}^{s}, x\partial_x + C_1(x,y)\partial_z, \partial_x, y\partial_y + C_2(x,y,z)\partial_z,$
$x^2\partial_x + sxy\partial_y + C_3(x,y,z)\partial_z, (x^m y^{n-j}\partial_z)_{j\to^* n}^{m\leq m_0+sj}\},$
$C_1(x,y) := \sum_{j=0}^{n} a_j(m_0+sj+1)x^{m_0+sj+1}y^{n-j},$
$C_2(x,y,z) := \frac{1}{s}\{(m_0+sn)z - \sum_{j=0}^{n} a_j(m_0+sj)x^{m_0+sj+1}y^{n-j}+$
$+a(m_0-s)y^{n+1}\},$
$C_3(x,y,z) := (m_0+sn)xz + \sum_{j=0}^{n} a_j x^{m_0+sj+2}y^{n-j}+$
$+\sum_{j=0}^{n} b_j x^{m_0+sj+1}y^{n-j} + a(m_0-s)xy^{n+1}.$

$\mathbf{ip}_{6,B}^{1} : \{\partial_x, y\partial_y + cx^{m_i+1}y^{m_i+2}\partial_z, x\partial_x + (m_i+1)z\partial_z, (x^m y^{2m_i+2-m_j}\partial_z)_{j\to^* h}^{m\leq m_j},$
$x^2 + \partial_x + xy\partial_y + C(x,y,z)\partial_z\},$
$C(x,y,z) := 2(m_i+1)xz + \frac{c}{m_i+2}x^{m_i+2}y^{m_i+2},\ m_{j+1} > m_j.$

$\mathbf{ip}_{6,B}^{2} : \{\partial_x, y\partial_y, x\partial_x + bz\partial_z, x^2\partial_x + xy\partial_y + 2bxz\partial_z, (x^m y^{2b-m_j})_{j\to^* h}^{m\leq m_j}\},$
$m_{j+1} > m_j.$

$\mathbf{ip}_{7,B} : \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y + (cx + c_1y^2)\partial_z, (x^l y^{-n}\partial_z)_{l\to^* m}^{n=l,\dots,m}\}.$

$\mathbf{ip}_{8,B} : \{\partial_x + \partial_y, x\partial_x + y\partial_y + \frac{m}{2}z\partial_z, x^2\partial_x + y^2\partial_y + C(x,y,z)\partial_z, (x^l\partial_z)_{l=0}^{m}\},$
$C(x,y,z) := mxz + c(y-x)^{\frac{m}{2}+1}.$

$\mathbf{ip}_{9,B} : \{(\Psi_{j,k}(x)y^{n_j-i}e^{c_jy}\partial_z)_{i\to^* n_j}^{k\to\sum_{u=0}^{i} l_u}\},\ j = 1,2,\dots,s.$

$\mathbf{ip}_{10,B}^{1} : \{\partial_y, y\partial_y + cz\partial_z, (\Psi_j(x)y^i\partial_z)_{i\to^* n}^{j\to l_i}\},\ l_n \leq l_{n-1} \leq \dots \leq l_0.$

$\mathbf{ip}^2_{10,B}: \{\partial_y, y\partial_y + c\partial_z, (\Psi_j(x)y^i\partial_z)^{j\to l_i}_{i\to^* n}\},\ l_n \le l_{n-1} \le \ldots \le l_0.$

$\mathbf{ip}_{11,B}: \{\partial_x, \partial_y + C(x,y)\partial_z, (x^m y^n e^{a_j x + b_j y}\partial_z)^{m\le m_{ij},\ n\le n_{ij}}_{j\to h_1,\ i\to k_j},$
$(x^p y^q e^{d_j y}\partial_z)^{p\le p_{ij},\ q\le q_{ij}}_{j\to h_2,\ i\to l_j}, (x^r y^s e^{c_j}\partial_z)^{r\le r_{ij},\ s\le s_{ij}}_{j\to h_3,\ i\to v_j}, (x^t y^u\partial_z)^{t\le t_j, u\le u_j}_{j\to h}\},$
$C(x,y) := \sum_{ij} h_{ij}x^{p_{ij}+1}y^{q_{ij}+1}e^{d_j y} + \sum_i k_i x^{t_i+1}y^{u_i},\ a_j, b_j, c_v, d_l \ne 0.$

$\mathbf{ip}_{12,B}: \{\partial_y, x\partial_x + y\partial_y, (y^l x^{a_i}\log^u x\partial_z)^{u\to^* u_{i_l}}_{l\to^* n,\ i\to i_l}\},$
$i_l \le i_{l-1} \le \ldots \le i_0,\ u_{i_l} \le u_{i_{l-1}} \le \ldots \le u_{i_0}.$

$\mathbf{ip}^1_{13,B}: \{\partial_x, \partial_y, y\partial_y + (cz + \bar{c}y^{u_1+1})\partial_z, (x^m y^n e^{a_j x}\partial_z)^{m\le m_{ij},\ n\le n_{ij}}_{j\to h_1,\ i\to k_j},$
$(x^t y^u\partial_z)^{t\le t_l,\ u\le u_l}_{l\to h}\},\ u_1 \ge u_2 \ge \ldots \ge u_h.$

$\mathbf{ip}^2_{13,B}: \{\partial_x, \partial_y + ax^{t_i+1}y^{u_i}\partial_z, y\partial_y + C(y,z)\partial_z, (x^m y^n e^{a_j x}\partial_z)^{m\le m_{ij}, n\le n_{ij}}_{j\to h_1,\ i\to k_j},$
$(x^t y^u\partial_z)^{t=t_l,\ u\le u_l}_{l\to h}\},$
$C(y,z) := (u_i+1)z + \bar{c}y^{u_1+1};\ u_1 \ge u_2 \ge \ldots \ge u_h.$

$\mathbf{ip}^1_{14,B}: \{\partial_y, g, \frac{1}{2}g_y, (\Psi_j(x)y^i\partial_z)^{j\to l_i}_{i\to^* n}\},$
$g := y^2\partial_y + (nz + 2cx)y\partial_z,\ c \in \{0,1\},\ l_n \le l_{n-1} \le \ldots \le l_0.$

$\mathbf{ip}^2_{14,B}: \{\partial_y, g, \frac{1}{2}g_y, (\Psi_j(x)\partial_z)_{j\to h}\},\ g := y^2\partial_y + 2(cx + \bar{c})y\partial_z,\ c\bar{c} = 0.$

$\mathbf{ip}^1_{15,B}: \{\partial_x, \partial_y, g, \frac{1}{2}g_y, (x^m y^l e^{a_i x}\partial_z)^{m\to^* m_{i_l}}_{l\to^* n,\ i\to i_l}, (x^m y^l\partial_z)^{l\le n}_{m\le m_0}, (x^m\partial_z)_{m\le m_1}\},$
$g := y^2\partial_y + (nyz + \frac{2a}{n+2}y^{n+2})\partial_z,$
$i_n \le i_{n-1} \le \ldots \le i_0;\ m_{i_n} \le m_{i_{n-1}} \le \ldots \le m_{i_0};\ a_i \ne 0,\ m_0 < m_1.$

$\mathbf{ip}^2_{15,B}: \{\partial_x, \partial_y, g, \frac{1}{2}g_y, (x^m e^{a_j x}\partial_z)^{m\le m_j}_{j\to h}, (x^m\partial_z)_{m\le m_0}\},$
$g := y^2\partial_y - (2yz + ax^{m_0-1})\partial_z,\ a_j \ne 0.$

$\mathbf{ip}^3_{15,B}: \{\partial_x, \partial_y, y\partial_y, y^2\partial_y + ay\partial_z, (x^m e^{a_j x}\partial_z)^{m\le m_j}_{j\to h}, (x^m\partial_z)_{m\le m_0}\}.$

$\mathbf{ip}^1_{16,B}: \{\partial_x, \partial_y, g, \frac{1}{2}g_y, x\partial_x + C(y,z)\partial_z, (x^m y^l\partial_z)^{l\le n}_{m\le m_0}, (x^m\partial_z)_{m\le m_1}\},$
$C(y,z) := b(z - \frac{2a}{n+2}y^{n+1}),\ g := y^2\partial_y + (nyz + \frac{2a}{n+2}y^{n+2})\partial_z,\ m_1 \ge m_0.$

$\mathbf{ip}^2_{16,B}: \{\partial_x, \partial_y, x\partial_x + (m_0+1)z\partial_z, g, \frac{1}{2}g_y, (x^m\partial_z)_{m\le m_0}\},$

$g := y^2\partial_y - (2yz + ax^{m_0+1})\partial_z.$

$\mathbf{ip}^3_{16,B}: \{\partial_x, \partial_y, y\partial_y, x\partial_x, y^2\partial_y + ay\partial_z, (x^m\partial_z)_{m\le m_0}\}.$

$\mathbf{ip}^4_{16,B}: \{\partial_x, \partial_y, y\partial_y, x\partial_x + bz\partial_z, y^2\partial_y, (x^m\partial_z)_{m\le m_0}\}.$

$\mathbf{ip}^1_{17,B}: \{\partial_x, \partial_y, g, \frac{1}{2}g_y, \bar{g}, \frac{1}{2}\bar{g}_x, (x^p y^q \partial_z)^{q\le n}_{p\le m}\},$
$g := y^2\partial_y + (nyz + \frac{2a}{n+2}y^{n+2})\partial_z,\ \bar{g} = x^2\partial_x + (mxz - \frac{2ma}{n+2}xy^{n+1})\partial_z.$

$\mathbf{ip}^2_{17,B}: \{\partial_x, \partial_y, g, \frac{1}{2}g_x, \bar{g}, \frac{1}{2}\bar{g}_y, (x^m\partial_z)_{m\le m_0}\},$
$g := x^2\partial_x + 2(m_0+1)xz\partial_z,\ \bar{g} := y^2\partial_y - (2yz + ax^{m_0+1})\partial_z.$

$\mathbf{ip}^3_{17,B}: \{\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x + ax\partial_z, y^2\partial_y + by\partial_z, (x^m\partial_z)_{m\le m_0}\}.$

$\mathbf{ip}^4_{17,B}: \{(y^i\partial_y)^2_{i=0}, \partial_x, x\partial_x + m_0 z\partial_z, x^2\partial_x + (2m_0 zx + ax^{m_0+1})\partial_z, (x^m\partial_z)_{m\le m_0}\}.$

$\mathbf{ip}_{18,B}: \{\partial_y + cz\partial_z, x\partial_y + bz\partial_z, (\Psi_l(x)\partial_y + b_l z\partial_z)^s_{l=1}, G\},$
$G := (y^{u-i-m_0-\ldots-m_s}\theta_{uj}(x)x^{m_0}\prod_{l=1}^s \Psi_l(x)^{m_l}\partial_z)^{i\le u-m_0-\ldots-m_s}_{u=n,\ldots,0,\ j\to q_u}.$

$\mathbf{ip}_{19,B}: \{\partial_y, y\partial_y + cz\partial_z, x\partial_y, (\Psi_l(x)\partial_y)^s_{l=1}, G\}$, G as in $\mathbf{ip}_{18,B}$.

$\mathbf{ip}_{20,B}: \{\partial_x, (e^{a_i x}(x^j\partial_y + \pi_{ij}\partial_z))^{j\to^* s_i}_{i\to q}, (x^m y^{n-t} e^{c_{j,n-t}x}\partial_z)^{m\to^* m_{j,n-t}}_{t\to^* n,\ j\to q_{n-t}}, G\},$
$G := (x^m y^{n-t-\sum h_l} e^{(c_{j,n-t}+\sum h_l a_{il})x}\partial_z)^{m\ge m_{j,n-t}+\sum h_l s_{i_l}}_{t\to^* n,\ j\to q_{n-t}};$
π_{ij} as in $\mathbf{ip}_{20,A}$, $\pi_{10} = 0$.

$\mathbf{ip}_{21,B}: \{\partial_x, y\partial_y + z\partial_z, \partial_y, (x^i\partial_y + \pi_i\partial_z)^s_{i=1}, (x^m y^{n-t} e^{c_{j,n-t}x}\partial_z)^{m\to^* m_{j,n-t}}_{t\to^* n,\ j\to q_{n-t}}, G,$
$(e^{a_i x}(x^j\partial_y + \pi_{ij}\partial_z))^{j\to^* s_i}_{i\to q}\}$, π_i, π_{ij} as in $\mathbf{ip}_{20,A}$,
$G := (x^m y^{n-t-h-\sum h_l} e^{(c_{j,n-t}+\sum h_l a_{il})x}\partial_z)^{m\ge m_{j,n-t}+hs+\sum h_l s_{i_l}}_{t\to^* n,\ j\to q_{n-t}}.$

3.5.5 Amaldis Groups of Type C

We repeat that the n-th Amaldi space group of type C, denoted by $\mathbf{ip}_{n,C}$, is represented by generators of the form

$$\xi_i(x)\partial_x + \eta_i(x,y)\partial_y + \zeta_i(x,y,z)\partial_z, \qquad i = 1,2,\ldots,l,$$
$$z\partial_z,\ \varphi_j(x,y)\partial_z, \qquad j = 1,2,\ldots,h>0,$$

where

$$\xi_i(x)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1,2,\dots,l$$

are the generators of the n-th Amaldi plane group, as listed in Section 3.5.1 ("Amaldis Imprimitive Plane Groups"). Furthermore, $\zeta_i(x,y,z)$ can be chosen as

$$\zeta_i(x,y,z) = \bar{\zeta}_i(x,y)z, \qquad i = 1,2,\dots,l.$$

$\mathbf{ip}_{1,C}$: $\{\partial_x, (x^i\partial_y)_{i=0}^{s}, x\partial_x + cy\partial_y, z\partial_z, (x^m y^{l-j}\partial_z)_{j\to^* l}^{m\to^* m_j}\}$,
$m_{j+1} \geq m_j + s$, size: $5 + l + s + \sum_{j=0}^{l} m_j$.

$\mathbf{ip}_{2,C}$: $\{\partial_x, (x^i\partial_y)_{i=0}^{s}, g, g_x - Cz\partial_z, z\partial_z, (x^m y^{l-j}\partial_z)_{j\to^* l}^{m\to^* m_0+sj}\}$, $C := m_0 + sl$;
$g := x^2\partial_x + sxy\partial_y + Cxz\partial_z$, size: $6 + l + s + m_0(l+1) + sl(l+1)/2$.

$\mathbf{ip}_{3,C}$: $\{\partial_x, (x^i\partial_y)_{i=0}^{s-1}, x\partial_x + (sy + x^s)\partial_y, z\partial_z, (x^m y^{l-j}\partial_z)_{j\to^* l}^{m\to^* m_j}\}$,
$m_{j+1} \geq m_j + s$, size: $4 + l + s + \sum_{j=0}^{l} m_j$.

$\mathbf{ip}_{4,C}$: $\{(x^i\partial_y)_{i=0}^{s}, y\partial_y, \partial_x, x\partial_x, z\partial_z, (x^m y^{l-j}\partial_z)_{j\to^* l}^{m\to^* m_j}\}$,
$m_{j+1} \geq m_j + s$, size: $6 + l + s + \sum_{j=0}^{l} m_j$.

$\mathbf{ip}_{5,C}$: $\{(x^i\partial_y)_{i=0}^{s}, y\partial_y, \partial_x, x\partial_x, g, z\partial_z, (x^m y^{l-j}\partial_z)_{j\to^* l}^{m\to^* m_0+sj}\}$,
g as in $\mathbf{ip}_{2,C}$, size: $7 + l + s + m_0(l+1) + sl(l+1)/2$.

$\mathbf{ip}_{6,C}$: $\{\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y + cxz\partial_z, z\partial_z, (x^m y^{c-m_j}\partial_z)_{j\to^* l}^{m\to^* m_j}\}$,
$m_{j+1} > m_j$, size: $6 + l + \sum_{j=0}^{l} m_j$.

$\mathbf{ip}_{7,C}$: $\{\partial_x, g, g_x, z\partial_z, (x^l y^{-n}\partial_z)_{l\to^* m}^{n=l,\dots,m}\}$,
$g := x^2\partial_x + xy\partial_y$, size: $m^2/2 + 3m/2 + 5$.

$\mathbf{ip}_{8,C}$: $\{(x^i\partial_x + y^i\partial_y + [i=2]mxz\partial_z)_{i=0}^{2}, z\partial_z, (x^i\partial_z)_{i\to^* m}\}$,
$m \geq 0$, size: $m + 5$.

$\mathbf{ip}_{9,C}$: $\{\partial_y, z\partial_z, (\Psi_{j,l}(x) y^m e^{c_j y}\partial_z)_{j\to s,\ m\to^* m_j}^{l\to l_{j,m}}\}$,
$l_{j,m_j} \leq l_{j,m_j-1} \leq \dots \leq l_{j,0}$, size: $2 + \sum_{j\to s}^{m\to^* m_j} l_{j,m}$.

$\mathbf{ip}_{10,C} : \{\partial_y, y\partial_y, z\partial_z, (\Psi_j(x)y^i\partial_z)_{i\to^* m}^{j\to l_i}\}$,
$l_m \le l_{m-1} \le \ldots \le l_0$, size: $3 + \sum_{i=0}^m l_i$.

$\mathbf{ip}_{11,C} : \{\partial_x, \partial_y, z\partial_z, (x^k y^l e^{a_m x + b_m y}\partial_z)_{m\to p,\ k\to^* k_m}^{l\to^* l_{m,k}}\}$,
$l_{m,k_m} \le l_{m,k_m-1} \le \ldots \le l_{m,0}$, size: $3 + p + \sum_{m=1}^p k_m + \sum_{m\to p}^{k\to^* k_m} l_{m,k}$.

$\mathbf{ip}_{12,C} : \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (y^l x^{a_i}(\log^u x)\partial_z)_{l\to^* m,\ i\to i_l}^{u\to^* u_{l,i}}\}$,
$i_l \le i_{l-1} \le \ldots \le i_0$, $u_{l,i} \le u_{l-1,i} \le \ldots \le u_{0,i}$, size: $3+\sum_{l=0}^m i_l + \sum_{l\to^* m}^{i\to i_l} u_{i,l}$.

$\mathbf{ip}_{13,C} : \{\partial_x, \partial_y, y\partial_y, z\partial_z, (x^i y^j e^{a_m x}\partial_z)_{m\to s,\ i\to^* i_m}^{j\to^* j_{m,i}}\}$,
$j_{m,i} \ge j_{m,i+1}$, size: $4 + s + \sum_{m=1}^s i_m + \sum_{m\to s}^{i\to^* i_m} j_{m,i}$.

$\mathbf{ip}_{14,C} : \{\partial_y, y\partial_y, y^2\partial_y + myz\partial_z, z\partial_z, (\Psi_j(x)y^i\partial_z)_{i\to^* m}^{j\to l}\}$, size: $4 + l + lm$.

$\mathbf{ip}_{15,C} : \{\partial_x, \partial_y, y\partial_y, y^2\partial_y + pyz\partial_z, z\partial_z, (x^m y^l e^{a_i x}\partial_z)_{i\to k,\ l\to^* p}^{m\to^* m_i}\}$,
size: $5 + (p+1)\sum_{i=1}^k m_i$.

$\mathbf{ip}_{16,C} : \{\partial_x, x\partial_x, \partial_y, y\partial_y, y^2\partial_y + lyz\partial_z, z\partial_z, (x^i y^j\partial_z)_{i\to^* m}^{j\to^* l}\}$, size. $7+l+m+lm$.

$\mathbf{ip}_{17,C} : \{\partial_x, \partial_y, x\partial_x, y\partial_y, (v^2\partial_v + lvz\partial_z)_{\{(x,m),(y,n)\}}^{(v,l)\in}, z\partial_z, (x^i y^j\partial_z)_{i\to^* m}^{j\to^* n}\}$,
size: $8 + m + n + mn$.

$\mathbf{ip}_{18,C} : \{\partial_y, (\Psi_k(x)\partial_y)_{k=0}^s, z\partial_z, (y^{u-i-\sum m_k}\theta_{u,j}(x)\prod_{k=0}^s \Psi_k(x)^{m_k}\partial_z)_{u\to^* l;\ j\to j_u}^{\sum_{k=0}^s m_k\to^* u}\}$,
$\Psi_0 \equiv x$, size: $3 + s + \sum_{u=0}^l j_u\binom{s+u+2}{u}$.

$\mathbf{ip}_{19,C} : \{y\partial_y\} \cup \mathbf{ip}_{18,C}$, same parameter ranges, size: $4 + s + \sum_{u=0}^l j_u\binom{s+u+2}{u}$.

$\mathbf{ip}_{20,C} : \{(x^m y^{l-\sum h_k} e^{(c_{l,j}+\sum h_k a_k)x}\partial_z)_{l\to^* t,\ j\to q_l,\ \mathbf{h}\in\{0,\ldots,l\}^q:\ \|\mathbf{h}\|_1\to^* l}^{m\to^* M_{l,j,\mathbf{h}}\ge m_{l,j}+\sum h_k s_k}$,
$\partial_x, z\partial_z, (x^j e^{a_i x}\partial_y)_{i\to q}^{j\to^* s_i}\}$, $a_1 \in \{0,1\}$, $q + \sum s_i \ge 2$,
size: $2 + q + \sum_{i=1}^q s_i + \sum_{l=0}^t q_l\binom{l+q}{q}\sum_{j=1}^{q_l}\sum_{\mathbf{h}\in\{0,\ldots,l\}^q}^{\|\mathbf{h}\|_1\to^* l} M_{l,j,\mathbf{h}}$.

$\mathbf{ip}_{21,C} : \{(x^m y^{l-\sum h_k} e^{(c_{l,j}+\sum h_k a_k)x}\partial_z)_{l\to^* t,\ j\to q_l,\ \mathbf{h}\in\{0,\ldots,l\}^{q+1}:\sum_{k=0}^q h_k\to^* l}^{m\to^* M_{l,j,\mathbf{h}}\ge m_{l,j}+\sum h_k s_k}$,
$\partial_x, y\partial_y, z\partial_z, (x^j e^{a_i x}\partial_y)_{i\to^* q}^{j\to^* s_i}\}$, $a_0 = 0$, $a_1 = 1$, $q \ge 0$, $q + \sum s_i \ge 0$,
size: $4 + q + \sum_{i=0}^q s_i + \sum_{l=0}^t q_l\binom{l+q+1}{q+1}\sum_{j=1}^{q_l}\sum_{\mathbf{h}\in\{0,\ldots,l\}^{q+1}}^{\|\mathbf{h}\|_1\to^* l} M_{l,j,\mathbf{h}}$.

3.5.6 Amaldis Groups of Type D

We repeat that the n-th Amaldi space group of type D, denoted by $\mathbf{ip}_{n,D}$, is represented by the generators of the form

$$\begin{aligned} \xi_i(x)\partial_x + \eta_i(x,y)\partial_y, &\qquad i = 1,2,\dots,l, \\ z^j\partial_z, &\qquad j = 0,1,2, \end{aligned}$$

where

$$\xi_i(x)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1,2,\dots,l$$

are the generators of the n-th Amaldi plane group, as listed in Section 3.5.1 ("Amaldis Imprimitive Plane Groups").

$\mathbf{ip}_{1,D}$: $\{(x^i\partial_y)_{i=0}^s, \partial_x, x\partial_x + cy\partial_y, (z^i\partial_z)_{i=0}^2\}$, $s \geq 1$, size: $s+6$.

$\mathbf{ip}_{2,D}$: $\{(x^i\partial_y)_{i=0}^s, \partial_x, g, g_x, (z^i\partial_z)_{i=0}^2\}$, $s \geq 1$, $g := x^2\partial_x + sxy\partial_y$, size: $s+7$.

$\mathbf{ip}_{3,D}$: $\{(x^i\partial_y)_{i=0}^{s-1}, \partial_x, x\partial_x + (sy + x^s)\partial_y, (z^i\partial_z)_{i=0}^2\}$, $s \geq 1$, size: $s+5$.

$\mathbf{ip}_{4,D}$: $\{(x^i\partial_y)_{i=0}^s, y\partial_y, \partial_x, x\partial_x, (z^i\partial_z)_{i=0}^2\}$, $s \geq 1$, size: $s+7$.

$\mathbf{ip}_{5,D}$: $\{(x^i\partial_y)_{i=0}^s, y\partial_y, (x^i\partial_x + [i=2]sxy\partial_y)_{i=0}^2, (z^i\partial_z)_{i=0}^2\}$, $s \geq 1$, size: $s+8$.

$\mathbf{ip}_{6,D}$: $\{y\partial_y, (x^i\partial_x + [i=2]xy\partial_y)_{i=0}^2, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{7,D}$: $\{\partial_x, g, g_x, (z^i\partial_z)_{i=0}^2\}$, $g := x^2\partial_x + xy\partial_y$.

$\mathbf{ip}_{8,D}$: $\{(x^i\partial_x + y^i\partial_y)_{i=0}^2, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{9,D}$: $\{\partial_y, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{10,D}$: $\{\partial_y, y\partial_y, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{11,D}$: $\{\partial_x, \partial_y, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{12,D}$: $\{\partial_y, x\partial_x + y\partial_y, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{13,D}$: $\{\partial_y, y\partial_y, \partial_x, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{14,D}$: $\{(y^i\partial_y)_{i=0}^2, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{15,D} : \{(y^i\partial_y)_{i=0}^2, \partial_x, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{16,D} : \{(y^i\partial_y)_{i=0}^2, \partial_x, x\partial_x, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{17,D} : \{(y^i\partial_y)_{i=0}^2, (x^i\partial_x)_{i=0}^2, (z^i\partial_z)_{i=0}^2\}$.

$\mathbf{ip}_{18,D} : \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^s, (z^i\partial_z)_{i=0}^2\}$, $s \geq 0$, size: $s+5$.

$\mathbf{ip}_{19,D} : \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^s, y\partial_y, (z^i\partial_z)_{i=0}^2\}$, $s \geq 0$, size: $s+6$.

$\mathbf{ip}_{20,D} : \{(x^j e^{a_i x}\partial_y)_{i\to l}^{j\to^* s_i}, \partial_x, (z^i\partial_z)_{i=0}^2\}$, $a_1 \in \{0,1\}$, $l \geq 1$, $\sum s_i + l \geq 2$,
size: $4 + l + \sum_{i=1}^l s_i$.

$\mathbf{ip}_{21,D} : \{(x^i\partial_y)_{i=0}^s, (x^j e^{a_i x}\partial_y)_{i\to l}^{j\to^* s_i}, y\partial_y, \partial_x, (z^i\partial_z)_{i=0}^2\}$,
$a_1 = 0$, $l \geq 0$, $\sum s_i + s + l \geq 1$, size: $6 + l + s + \sum_{i=1}^l s_i$.

Chapter 4

Differential Invariants of Order Two

In this chapter, we list differential invariants of order two of many of the point transformation groups listed in Chapter 3 ("The Space Point Groups"). These invariants of order two were obtained by first prolongating the corresponding generators to order two as described in Subsection 2.2.1 ("Extended Infinitesimal Transformations"). They form a fundamental system for the corresponding system of differential invariants, which is obtained by interpreting the prolongations as a system of linear PDEs in the variables $x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}$. This system was usually solved by iterated narrowing transformations as described in Subsection 2.2.2 ("Solving Systems of Linear Homogeneous PDEs").

In this chapter we treat all classes of space groups except Amaldis groups of type B. All groups considered have less than eight parameters. Those groups are guaranteed to have invariant bases. The next chapter deals with groups that have more than seven parameters.

In Section 4.1 ("Invariants of Lie's Plane Groups"), we present Lie's results concerning the groups of the plane. In Section 4.2 ("Definition of Subexpressions") we define expressions that simplify and shorten the presentation of the invariant list. Finally, in Section 4.3 ("List of Invariants According to Derived Series") the complete list of invariants sorted lexicographically according to their derived series is presented.

4.1 Invariants of the Groups of the Plane

We present Lie's results on differential invariants for the groups of the x, y-plane with y depending on x. For each plane group, we give the two lowest invariants Φ_1 and Φ_2. The higher order ones may then be obtained recursively by differentiation according to

$$\Phi_j = \frac{\Phi'_{j-1}}{\Phi'_1}, \quad j \geq 3. \tag{4.1}$$

The lower equations of the group are functions of the irreducible factors of the determinant

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1^{(1)} & \zeta_1^{(r-2)} \\ \xi_2 & \eta_2 & \zeta_2^{(1)} & \zeta_2^{(r-2)} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_r & \eta_r & \zeta_r^{(1)} & \zeta_r^{(r-2)} \end{vmatrix}.$$

In various cases, the abbreviation $w = \frac{y'''}{y'} - \frac{3}{2}(\frac{y''}{y'})^2$ for the Schwarzian derivative is used. This listing is taken from [10], it basically corresponds to the listing given by Lie [6], with some minor improvements included.

g₁: $\Phi_1^{(1)} = \frac{(3y''y^{(4)} - 5y'''^2)^3}{y''^8}$, $\Phi_2^{(1)} = \frac{3y''^2y^{(5)} - 15y''y'''y^{(4)} + \frac{40}{3}y'''^3}{y''^4}$, $\Delta^{(1)} = 9y''^3$.

g₂: $\Phi_1^{(2)} = \frac{(3y''^2y^{(5)} - 15y''y'''y^{(4)} + \frac{40}{3}y'''^3)^2}{(3y''y^{(4)} - 5y'''^2)^3}$,

$\Phi_2^{(2)} = \frac{3y''^3y^{(6)} - 21y''^2y'''y^{(5)} + 35y''y'''^2y^{(4)} - \frac{35}{3}y'''^4}{(3y''y^{(4)} - 5y'''^2)^2}$,

$\Delta^{(2)} = 2y''^2(5y'''^2 - 3y''y^{(4)})$.

g₃: $\Phi_1^{(3)} = \frac{u^3}{\rho_5^8}$, $\Phi_2^{(3)} = \frac{v}{\rho_5^3}$, $\Delta^{(3)} = -2y''(9y''^2y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3)^2$,

where

$\rho_4 = 3y''y^{(4)} - 4y'''^3$, $\rho_5 = 3y''^2y^{(5)} - 15y''y'''y^{(4)} + \frac{40}{3}y'''^3$,

$\rho_6 = 3y''^2y^{(6)} - 24y''^2y'''y^{(5)} + 60y''y'''^2y^{(4)} - 40y'''^4$,

$$\rho_7 = 9y''^4y^{(7)} - 105y''^3y'''y^{(6)} + 420y''^2y'''^2y^{(5)} - 7000y''y'''^3y^{(4)} + \tfrac{1120}{3}y'''^5,$$

$$\rho_8 = 27y''^5y^{(8)} - 48y'''\rho_7 - 840y'''^2\rho_6 - 2240y'''^3\rho_5 - 2800y'''^4\rho_4 - \tfrac{2240}{3}y'''^6,$$

$$u = 2\rho_5\rho_7 - 35\rho_4\rho_5^2 - 7(\rho_6 - \tfrac{5}{3}\rho_4^2)^2,$$

$$v = \rho_5(\rho_8 - 84\rho_4\rho_6 + \tfrac{245}{3}\rho_4^3) - 12(\rho_7 - \tfrac{35}{2}\rho_4\rho_5)(\rho_6 - \tfrac{5}{3}\rho_4^2) + \tfrac{28}{\rho_5}(\rho_6 - \tfrac{5}{3}\rho_4^2)^3.$$

$\mathbf{g}_4$**:** $\Phi_1^{(4)} = x,\ \Phi_2^{(4)} = \frac{y''}{y'},\ \Phi_3^{(4)} = \frac{y'''}{y'},\ \Delta^{(4)} = 0.$

$\mathbf{g}_5$**:** $\Phi_1^{(5)} = \frac{y''}{y'},\ \Phi_2^{(5)} = \frac{y'''}{y'},\ \Delta^{(5)} = y'.$

$\mathbf{g}_6$**:** $\Phi_1^{(6)} = \frac{y'y'''}{y''^2},\ \Phi_2^{(6)} = \frac{y'^2y^{(4)}}{y''^3},\ \Delta^{(6)} = -y'y''.$

$\mathbf{g}_7$**:** $\Phi_1^{(7)} = \frac{y''^{c-1}}{y'^{c-2}},\ \Phi_2^{(7)} = \frac{y'''^{c-1}}{y'^{c-3}},\ \Delta^{(7)} = (c-1)y',\ c \neq 1.$

$\mathbf{g}_8$**:** $\Phi_1^{(8)} = x,\ \Phi_2^{(8)} = w,\ \Delta^{(8)} = 0.$

$\mathbf{g}_9$**:** $\Phi_1^{(9)} = w,\ \Phi_2^{(9)} = w',\ \Delta^{(9)} = 2y'^3.$

$\mathbf{g}_{10}$**:** $\Phi_1^{(10)} = \frac{((x-y)y''+2y'(y'+1))^2}{y'^3},\ \Phi_2^{(10)} = \frac{(x-y)^2y'''+6(x-y)(y'+1)y''}{y'^2} + \frac{6(y'^2+4y'+1)}{y'},$

$\Delta^{(10)} = 2(y-x)^2y'.$

$\mathbf{g}_{11}$**:** $\Phi_1^{(11)} = \frac{w'^2}{w^3},\ \Phi_2^{(11)} = \frac{w''}{w^3},\ \Delta^{(11)} = 4y'^2(y'y''' - \frac{3}{2}y''^2).$

$\mathbf{g}_{12}$**:** $\Phi_1^{(12)} = \frac{4ww''-5w'^2}{w^3},\ \Phi_2^{(12)} = \frac{(4w^2w'''-18ww'w''+15w'^3)^2}{w^9},$

$\Delta^{(12)} = 4y'^2(y'y''' - \frac{3}{2}y''^2).$

$\mathbf{g}_{13}$**:** $\Phi_1^{(13)} = y''y^3,\ \Phi_2^{(13)} = y'''y^5 + 3y''y'y^4,\ \Delta^{(13)} = y^2.$

g$_{14}$**:** $\Phi_1^{(14)} = \frac{(yy'''+3y'y'')^2}{yy''^3}$, $\Phi_2^{(14)} = \frac{3yy''y^{(4)}-4yy'''^2}{yy''^3}$, $\Delta^{(14)} = 2y^2y''$.

g$_{15}$**:** $\Phi_1^{(15)} = x$, $\Phi_2^{(15)} = D$, $\Delta^{(15)} = 0$, where

$$D = \begin{vmatrix} \Phi_1'' & \Phi_1''' & \cdots & \Phi_1^{(r+2)} \\ \vdots & \vdots & \vdots & \\ \Phi_r'' & \Phi_1''' \ldots & \Phi_r^{(r+2)} & \\ y'' & y''' & \cdots & y^{(r+2)} \end{vmatrix}.$$

g$_{16}$**:** $\Phi_1^{(16)} = x$, $\Phi_2^{(16)} = (\log D)'$, $\Delta^{(16)} = 0$, D as in **g**$_{15}$.

g$_{17}$**:** $\Phi_1^{(17)} = c_0y + c_1y' + \ldots + c_ry^{(r)}$, $\Phi_2^{(17)} = (\Phi_1^{(17)})'$.

g$_{18}$**:** $\Phi_1^{(18)} = \frac{D_1}{D}$, $\Phi_2^{(18)} = \frac{D_2}{D}$, with D,D_1, and D_2 as above, if the proper values for the Φ_k are substituted. The invariants are of order $l+\sum\rho_k+1$ and $l+\sum\rho_k+2$ respectively.

g$_{19}$**:** For c arbitrary: $\Delta^{(19)} = (c-r)y^{(r)}$.

For $c \neq r$: $\Phi_1^{(19)} = \frac{y^{(r+1)^{c-r}}}{y^{r^{c-r-1}}}$, $\Phi_2^{(19)} = \frac{y^{(r+2)^{c-r}}}{y^{r^{c-r-2}}}$.

For $c = r$: $\Phi_1^{(19)} = y^{(r)}$, $\Phi_2^{(19)} = \frac{y^{(r+2)}}{y^{(r+1)^2}}$.

g$_{20}$**:** $\Phi_1^{(20)} = y^{(r+1)}e^{y^{(r)}/r!}$, $\Phi_2^{(20)} = y^{(r+2)}e^{2y^{(r)}/r!}$, $\Delta^{(20)} = \prod_{k=1}^r k!$.

g$_{21}$**:** $\Phi_1^{(21)} = \frac{y^{(r)}y^{(r+2)}}{y^{(r+1)^2}}$, $\Phi_2^{(21)} = \frac{{y^{(r)}}^2y^{(r+3)}}{y^{(r+1)^3}}$, $\Delta^{(21)} = y^{(r)}y^{(r+1)}$.

g$_{22}$**:** $\Phi_1^{(22)} = \frac{v_1^{r+1}}{y^{(r)^{2(r+3)}}}$, $\Phi_2^{(22)} = \frac{v_2^{r+1}}{y^{(r)^{3(r+3)}}}$, $\Delta^{(22)} = {y^{(r)}}^2$, where

$v_1 = (r+1)y^{(r)}y^{(r+2)} - (r+2)^{(r+1)^2}$,

$v_2 = (r+1)^2{y^{(r)}}^2y^{(r+3)} - 3(r+1)(r+3)y^{(r)}y^{(r+1)}y^{(r+2)} +$

$+2(r+2)(r+3)y^{(r+1)^3}.$

$\mathbf{g}_{23}$: $\Phi_1^{(23)} = \frac{v_2^2}{v_1^3}$, $\Phi_2^{(23)} = \frac{v_3}{v_1^2}$, $\Delta^{(23)} = y^{(r)}((r+2)y^{(r+1)^2} - (r+1)y^{(r)}y^{(r+2)})$,
where v_1 and v_2 are as in $\mathbf{g}_{22}$ and

$$v_3 = (r+1)^3 y^{(r)^3} y^{(r+4)} - 4(r+1)^2(r+4)y^{(r)^2}y^{(r+1)}y^{(r+3)} +$$
$$+6(r+1)(r+3)(r+4)y^{(r)}y^{(r+1)^2}y^{(r+2)} -$$
$$-3(r+2)(r+3)(r+4)y^{(r+1)^4}.$$

$\mathbf{g}_{24}$: $\Phi_1^{(24)} = y'$, $\Phi_2^{(24)} = \frac{y'''}{y''^2}$, $\Delta^{(24)} = 0$.

$\mathbf{g}_{25}$: $\Phi_1^{(25)} = y'$, $\Phi_2^{(25)} = xy''$, $\Delta^{(25)} = -x$.

$\mathbf{g}_{26}$: $\Phi_{m-1}^{(26)}(y', y'', \ldots, y^{(m)})$ for $m \geq 2$, $\Delta^{(26)} = 1$.

$\mathbf{g}_{27}$: $\Phi_{m-1}^{(27)}(x, y', y'', \ldots, y^{(m)})$ for $m \geq 1$, $\Delta^{(27)}$ not defined.

4.2 Definition of Subexpressions

In order to give a compact presentation of the invariant list, we define the following expressions for each class of groups. Their order is mainly determined by their appearance in the invariant list. For example if H_i contains a subexpression, that appears as H_j at some other place in the subexpression list for type A, we refer to H_j in order to shorten the presentation (e.g. $H_2 := H_{35}/z_y^2$). The recursion depth hereby usually is one, in a few cases it is two.

We use the convention that free variables in the subexpression S match the parameters of the group G, in case that S appears in the invariants of G.

Example: An entry in the invariant list of the form $\mathbf{ip}_{6,C}[\mathbf{m} = [1]]$: I_{29}, where I_{29} is defined in the subexpression list as

$$I_{29} := \frac{y^2 z_{yy} - c^2 z + (2m_0 + 1)cz - m_0(m_0+1)z}{yz_y - cz + m_0 z}$$

indicates that

$$\frac{y^2 z_{yy} - c^2 z + 3cz - 2z}{yz_y - cz + z}$$

is a differential invariant of $\mathbf{ip}_{6,C}[\mathbf{m} = [1]]$. Note that $m_0 = 1$ since $\mathbf{m} = [m_0]$.

4.2.1 Subexpressions for Lie's Space Groups

For Lie's primitive groups we define

$$F_1 := 1 + z_x^2 + z_y^2,\ F_2 := z_{xy}^2 - z_{xx} z_{yy},\ F_3 := z_x^2 z_{yy} + z_y^2 z_{xx} - 2 z_x z_y z_{xy}.$$

For Lie's imprimitive groups we define

$$\begin{aligned}
G_1 &:= z_x + z z_y,\ G_2 := z_{xy}^2 - z_{xx} z_{yy},\ G_{2,1} := G_2 - z_{xy},\\
G_3 &:= z_x^2 z_{yy} + z_y^2 z_{xx} - 2 z_x z_y z_{xy},\\
G_{3,1} &:= G_3 + z_x z_y + y(2 z_y z_{xy} - 2 z_x z_{yy} - z_y) + y^2 z_{yy},\\
G_4 &:= z_x z_{xy} - z_y z_{xx} + z(z_x z_{yy} - z_y z_{xy}),\\
G_5 &:= z_{xx} + z^2 z_{yy} - 2 z_x z_y + 2z(z_{xy} - z_y^2).
\end{aligned}$$

4.2.2 Subexpressions for Groups of Type A

$$H_1 := (H_{48})_{a=1+c/t},\ H_2 := H_{35}/z_y^2,\ H_3 := z_y^{c-2}H_{34},\ H_4 := z_y^{c-3}H_{18},$$
$$H_5 := z^{t/(c-t)}z_y,\ H_6 := z^{(c+t)/(c-t)}z_{yy},\ H_7 := z^{(t+1)/(c-t)}H_{29}/z_y,$$
$$H_8 := H_{29}/z_y^{(2t+1)/t},\ H_9 := z_x^2 z_{yy} + z_y^2 z_{xx} - 2z_x z_y z_{xy},$$
$$H_{10} := z_{yy}(xz_x + 2zz_y) - z_{xy}^2,\ H_{11} := (H_{48})_{a=2},\ H_{12} := cH_{33} + H_{34}/z_y,$$
$$H_{13} := (H_{39})_{a=1},\ H_{14} := H_{18}/z_y^2 + cH_{33}(H_{39})_{a=2},\ H_{15} := z_y(H_{51})_{a=t},$$
$$H_{16} := z_{yy}(H_{51})_{a=2t},\ H_{17} := H_{29}(H_{51})_{a=t+1}/z_y,\ H_{18} := z_{xx} + 2zz_{xy} + z^2 z_{yy},$$
$$H_{19} := H_{29}/z_y^{2+1/t},\ H_{20} := (z_{xx}z_{yy} - z_{xy}^2)/(z_y z_{yy}) + 2z + cH_{33},$$
$$H_{21} := (sH_{35} + z_y^2)/z_{yy}^{4/(s-2)},\ H_{22} := H_{27}/z_{yy}^{5/(2s+1)},$$
$$H_{23} := z^{(s-1)^2/\{(s-1)^2+1\}}(H_{40})_{a=-3}/z_y^{(2s+2)/(s+2)},$$
$$H_{24} := (4cz^2 H_{29} + (H_{40})_{a=2s+1})/(z^{(s-1)/s}z_y^{(9-s^2)/(4-s)}),$$
$$H_{25} := \{8cz^2(2czH_9 + H_{29}) + (H_{40})_{a=1}\}/z_y^4,\ H_{26} := (H_{48})_{a=(s+t)/t},$$
$$H_{27} := H_{18} + (5-s^2)z_y H_{34},\ H_{28} := H_{34} + 2H_{33},\ H_{29} := z_x z_{yy} - z_y z_{xy},$$
$$H_{30} := (tH_{29} + sz_{yy}H_{33})/z_y^{(s+t)/t},\ H_{31} := z_{yy}(z_{xx}z_{yy} - z_{xy}^2) + 2zz_y + 6H_{33},$$
$$H_{32} := \{6H_{33}(3z_{yy} + 2H_{29}) + 4(z_x^2 z_{yy} + z_y^2 z_{xx}) + 8(zz_y^3 - z_x z_y z_{xy})\}/z_y^{3/2},$$
$$H_{33} := \log(z_y),\ H_{34} := z_x + zz_y,\ H_{35} := z_{xy} + zz_{yy},\ H_{36} := (cz_x + zz_y)H_{47},$$
$$H_{37} := (c^2 z_{xx} + z^2 z_{yy} + 2czz_{xy})H_{47},$$
$$H_{38} := H_{47}(z_{xx} + 2zz_y/c - 2fz_{xy} + f^2 z_{yy})_{f=z_x/z_y},$$
$$H_{39} := (cz_{yy}H_{33} + aH_{35})/z_y^2,\ H_{40} := zz_{yy} + az_y^2,\ H_{41} := zH_{11},$$
$$H_{42} := H_9(H_{48})_{a=2s+1},\ H_{43} := H_{18}(H_{48})_{a=4},\ H_{44} := H_{34}(H_{48})_{a=3},$$
$$H_{45} := \{z_{yy}(z_{xx} + 2zz_y) - z_{xy}^2\}/z_y^3,\ H_{46} := z_y^2 + sH_{35},\ H_{47} := (H_{51})_{a=2},$$
$$H_{48} := z_{yy}/z_y^a,\ H_{49} := yz_y,\ H_{50} := y^2 z_{yy},\ H_{51} := \exp(az/c),$$
$$H_{52} := -yH_{29} - (1+2a)z_x z_y,\ H_{53} := 2z + yz_y,$$
$$H_{54} := 4yz(z_y z_{xx} - z_x z_{xy}) + 2z(2zz_{xx} - 3z_x^2),\ H_{55} := z + xz_x,$$
$$H_{56} := 2z_x^2 - zz_{xx},\ H_{57} := 1 + cz_y,\ H_{58} := czz_{xy} - 2z_x H_{57},$$
$$H_{59} := c^2(H_{40})_{a=-2} - 6cz_y - 4,\ H_{60} := z_x^{c-1}(zz_{xy} - z_x z_y)/z^c,$$
$$H_{61} := (z_x/z)^{2c}(H_{40})_{a=-1/2},\ H_{62} := 2c(zz_{xx}/z_x^2 - 1) + 1,$$
$$H_{63} := \varphi_2'(x)\{z_{yy}(z_{xx} + zz_y\Psi_1''(x)) - z_{xy}^2\} -$$
$$- H_{29}\{\varphi_2''(x) + z_y(\Psi_1''(x)\varphi_2(x) - \Psi_2''(x))/\varphi_2'(x)\},$$
$$H_{64} := H_9 + zz_y^3\Psi_1''(x),\ H_{65} := z_y^2 + 2z_{xy},\ H_{66} := \log((H_{48})_{a=2}).$$

4.2.3 Subexpressions for Groups of Type C

$$\begin{aligned}
I_1 &:= z_x/z_y,\ I_2 := z_{xx}/z_x,\ I_3 := z_{xy}/z_x,\ I_4 := z_{yy}/z_x,\ I_5 := z_{xx}/z_y,\\
I_6 &:= z_{xy}/z_y,\ I_7 := z_{yy}/z_y,\ I_8 := z_{xx}z_{yy}/z_{xy}^2,\ I_9 := z_x z_{yy} - z_y z_{xy},\\
I_{10} &:= z_x^2 z_{yy} + z_y^2 z_{xx} - 2z_x z_y z_{xy},\ I_{11} := z_{xx}z_{yy} - z_{xy}^2,\ I_{12} := z_{xy}/z_{xx},\\
I_{13} &:= z_{yy}/z_{xx},\ I_{14} := z_{xy}/z_{yy},\ I_{15} := e^{-y}/I_i,\ I_{16} := e^{(a-i)y}I_i,\\
I_{17} &:= I_i/(z_y^{a-i}e^{I_1}),\ I_{18} := I_{14} - \log(I_7),\ I_{19} := (z_{xx} - Az_{xy})/(z_y e^A),\ \text{where}\\
A &:= f(I_{14}e^{I_{14}}),\ f \text{ analytic at 0 with } f(x)e^{f(x)} \equiv x \text{ (“Lambert wavef.”)}.\\
I_{20} &:= I_9/(z_y^2 I_7^{1/s}),\ I_{21} := I_i/(z_y^{3-m_0}I_7^{m_0}) - \log(I_7),\ I_{22} := 1/(I_{12}I_{14}),\\
I_{23} &:= z_x z_y + yI_9,\ I_{24} := (2z_y + yz_{yy})/y,\ I_{25} := z_x^2 - y^2 I_{11} + 2yI_{30},\\
I_{26} &:= (I_{27,x,x,0}^{(0,0)})^{1/3},\ I_{27,a,b,c}^{(d,e)} := \{(x-y)z_{ab} + cz_y\}/(z_x^d z_y^e),\\
I_{28} &:= (x-y)\{(x-y)z_{xy}z_{yy} - z_y(2z_{xy} + mz_{yy})\}/z_y^2,\\
I_{29} &:= (y^2 z_{yy} - c^2 z + (2m_0+1)cz - m_0(m_0+1)z)/(yz_y - cz + m_0 z),\\
I_{30} &:= z_x z_{xy} - z_y z_{xx},\ A := y^2 I_{10} + 2cyzI_{30} + c(1-c)zz_x^2 + (cz)^2 z_{xx},\\
B &:= yI_9 + (1-c)z_x z_y + czz_{xy},\ I_{31} := A(cz - yz_y)/(yB)^2,\\
m^+ &:= m_1 + m_0,\ m^- := m_1 - m_0,\ m^* := m_1 m_0,\ (m_i^+ := m_i(m_i+1))_{i=1}^2,\\
M_1 &:= m^-(c^2 - cm^+ + m^*),\ M_2 := m_1^+ - m_0^+ - 2m^- c,\\
I_{32} &:= y^{2c-m^+-3}[M_1 z + M_2 yz_y + m^- y^2 z_{yy}],\\
I_{33} &:= y^{2-c}z_{xx}I_{32} + (2c - c^2 - 1)y^{c-2}z_x^2 + 2(c-1)y^{c-1}z_x z_{xy} - y^c z_{xy}^2,\\
I_{34} &:= c_1 z - z_y,\ I_{35} := c_1^2 z - z_{yy},\ I_{36}(v_1, v_2, F_1, F_2) := v_1 F_2 - v_2 F_1,\\
I_{37} &:= I_{36}(z, z_x, \Psi_{1,1}(x), \Psi_{1,1}'(x)),\ I_{38} := I_{36}(c_1 z, z_{xy}, \Psi_{1,1}(x), \Psi_{1,1}'(x)),\\
I_{39} &:= I_{36}(z, z_{xx}, \Psi_{1,1}(x), \Psi_{1,1}''(x)),\ I_{40} := c_1 z_x - z_{xy},\ I_{41}(F_1, F_2) :=\\
&:= \det\left(\begin{bmatrix} z & F_1 & F_2 \\ z_x & F_1' & F_2' \\ z_{xx} & F_1'' & F_2'' \end{bmatrix}\right),\ I_{42} := I_{41}(\Psi_{1,1}(x), \Psi_{1,2}(x)),\\
I_{43} &:= I_{36}(z_y, z_{xy}, \Psi_{1,1}(x), \Psi_{1,1}'(x)),\ I_{44} := c_1^2 z - 2c_1 z_y + z_{yy},\\
I_{45} &:= c_1 I_{37} - I_{43},\ I_{46} := I_7/z_y,\ I_{47} := I_{36}(z, z_x, \Psi_1(x), \Psi_1'(x)),\\
I_{48} &:= I_{36}(z, z_{xx}, \Psi_1(x), \Psi_1''(x)),\ I_{49} := I_{41}(\Psi_1(x), \Psi_2(x)),\\
I_{50} &:= I_{36}(z_y, z_{xy}, \Psi_1(x), \Psi_1'(x)),\ I_{51} := zz_{xy} - z_x z_y,
\end{aligned}$$

$$I_{52} := z^2(mzz_{yy} - (m-1)z_y^2)/I_{51}^2,\ I_{53} := (a_1^2 z - z_{xx})/I_{68},$$
$$I_{54} := a_1^2 z - 2a_1 z_x + z_{xx},\ I_{55} := (\prod a_i)z - (\sum a_i)z_x + z_{xx},$$
$$I_{56} := zI_{69},\ I_{57} := I_9/z_y^2,\ I_{58} := I_{10}/(zz_y^2),$$
$$I_{59} := I_{36}(zz_{yy}, I_9, \theta_{0,1}(x), \theta_{0,1}(x)')/z_y^2,\ \text{in } I_{60} \text{ note } \theta_{0,1} = \theta_{0,1}(x):$$
$$I_{60} := I_{36}(I_9, I_{10}, \theta_{0,1}^2, 2z\theta_{0,1}\theta_{0,1}') + I_{36}(zz_y^2, z^2 z_{yy}, (\theta_{0,1}')^2, \theta_{0,1}\theta_{0,1}''),$$
$$I_{61} := I_{62}(\theta_{0,1}(x), \theta_{0,2}(x)),\ I_{62}(F_1,F_2) := \det\left(\begin{bmatrix} zz_{yy} & F_1 & F_2 \\ I_9 & F_1' & F_2' \\ I_{11} & F_1'' & F_2'' \end{bmatrix}\right),$$
$$I_{63} := z_y/z,\ I_{64} := z_{yy}/z,\ I_{65} := I_9/(zz_y),\ I_{66} := I_{58},\ \text{note } \theta_{0,1} = \theta_{0,1}(x):$$
$$I_{67} := I_{36}(I_9, I_{11}, \theta_{1,1}^2, 2\theta_{1,1}\theta_{1,1}') + I_{36}(zz_{yy}, 2zz_{yy} - z_y^2, (\theta_{1,1}')^2, \theta_{1,1}\theta_{1,1}''),$$
$$I_{68} := a_1 z - z_x,\ I_{69} := z_{yy}/z_y^2,\ I_{70} := I_{68}I_{69},\ I_{71} := I_{54}I_{69},$$
$$I_{72} := z_{yy}I_{68}/I_{73}^2,\ I_{73} := (I_{77})_{M=1},\ I_{74} := I_{55}I_{69},$$
$$I_{76} := z_{yy}I_{55}/I_{73}^2,\ I_{76} := z_{yy}I_{54}/I_{73}^2,\ I_{77} := a_M z_y - z_{xy},\ I_{78} := a_1 z - xz_x,$$
$$I_{79} := xz_{yy}/I_{78},\ I_{80} := x^2 z_{yy}/I_{78},\ I_{81} := a_1(1-a_1)z + x^2 z_{xx},$$
$$I_{82} := xI_6,\ I_{83} := xI_7,\ I_{84} := x^2 z_{xx} + (1-2a_1)xz_x + a_1^2 z,$$
$$I_{85} := x^2 z_{xx} + (\prod a_i)z + (1 - \sum a_i)xz_x,\ I_{86} := a_1 z_y - xz_{xy},$$
$$I_{87} := x^{2a_1-1}z_{xx} + (1-2a_1)x^{2a_1-2}z_x + a_1^2 x^{2a_1-3}z.$$

$$D := [[z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}], [1, a_i, b_i, a_i^2, a_i b_i, b_i^2]_{i=1}^5],$$
$$d_J := \det([D_{i,j}]_{j=J}^{i\to n}),\ \text{where } J := j_1, \ldots, j_m.$$

For $\mathbf{ip}_{9,C}[\mathbf{L} = [[(l_{j,m})_{m\to^* m_j}]_{j\to s}]]$ we define E and e as

$$E(\mathbf{L}) := [[z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}], Pr^{(2)}(y^m \Psi_{j,l}(x)e^{c_j x}\partial_z)_{j\to s,\ m\to^* m_j}^{l\to l_{j,m}}],\ \text{where}$$
$$Pr^{(2)}(\cdot) := [y^m\Psi_{j,l}, y^m\Psi_{j,l}', my^{m-1}\Psi_{j,l} + c_j y^m \Psi_{j,l}, y^m\Psi_{j,l}'', my^{m-1}\Psi_{j,l}' +$$
$$+ c_j y^m \Psi_{j,l}', m(m-1)y^{m-2}\Psi_{j,l} + 2c_j m y^{m-1}\Psi_{j,l} + c_j^2 y^m \Psi_{j,l}],$$
$$e_J(\mathbf{L}) := pp(\det([E_{i,j}]_{j=J}^{i\to k}), [z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}]),\ \text{where } J := j_1, \ldots, j_k$$

and $pp(\cdot, [z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}])$ denotes the primitive part of a (linear) polynomial in $[z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}]$.

4.2.4 Subexpressions for Groups of Type D

$J_0 := z_y/z_x,\ J_{0,1} := z_x/z_y,$
$J_1 := z_x z_{xy} - z_y z_{xx},\ J_{1,1} := J_1/z_x^2,\ J_{1,2} := J_1/(z_x z_y),$
$J_2 := z_x z_{yy} - z_y z_{xy},\ J_{2,1} := J_2/z_y^2,\ J_{2,2} := J_2/(z_x z_y),\ J_{2,3} := yJ_2 + z_x z_y,$
$J_3 := z_x^2 z_{yy} - z_y^2 z_{xx},\ J_{3,1} := J_3/z_x^3,\ J_{3,2} := J_3/(z_x z_y^2),$
$J_4 := z_x^2 z_{yy} + z_y^2 z_{xx} - 2z_x z_y z_{xy},\ J_{4,1} := J_4/z_y^3.$

4.3 List of Differential Invariant Bases

Here we list the groups, represented by their Lie algebras as usual, and their invariants according to the derived series of the Lie algebra. Since derived series are invariant under point transformations, this listing simplifies the identification of a given group. Groups with the same derived series are ordered according to their type, their number and their parameters. The generator lists are ordered according to the coefficients of $\partial_x,\ \partial_y,\ \partial_z$. The scheme of presentation is

$$< \text{group identifier} > = < \text{generatorlist} > : < \text{invariant basis} > .$$

The notation for derived series can be found in Section 2.4 ("Basic Notions for Lie Algebras").

4.3.1 Groups with Seven Generators

Derived Series (7)

$\mathbf{ip}_2[l=2] = \{x\partial_y, x\partial_x - y\partial_y, y\partial_x, \partial_x, Z(z)\partial_x, \partial_y, Z(z)\partial_y\}$:
$z,\ (Z'G_2 - Z''G_3)/(Z'G_3^{4/3})$.

$\mathbf{ip}^1_{2,A}[s=3] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, x^3\partial_y + 3x^2\partial_z, \partial_x,$
$2x\partial_x + 3y\partial_y + z\partial_z, x^2\partial_x + 3xy\partial_y + (xz+3y)\partial_z\}$: H_{21}.

$\mathbf{ip}^2_{2,A}[s=3] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, \partial_x, 2x\partial_x + 3y\partial_y + z\partial_z,$
$x^2\partial_x + 3xy\partial_y + z(x+cz^2)\partial_z\}$: $H_{23},\ H_{24}$.

$\mathbf{ip}_{5,A}[s=2] = \{\partial_x, \partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, x\partial_x - z\partial_z, y\partial_y + z\partial_z,$

$x^2\partial_x + 2xy\partial_y + 2y\partial_z\}$: $z_{yy}H_{27}/H_{46}$.

$\mathbf{ip}_{6,C}[\mathbf{m} = [1]] = \{\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y + cxz\partial_z, z\partial_z, y^{c-1}\partial_z, xy^{c-1}\partial_z\}$: I_{29}.

$\mathbf{ip}_{8,C}[m = 2] = \{\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y + 2xz\partial_z, z\partial_z, \partial_z, x\partial_z, x^2\partial_z\}$: I_{28}.

$\mathbf{ip}_{14,C}[m = 2, l = 1] = \{\partial_y, y\partial_y, y^2\partial_y + 2(y-1)z\partial_z, z\partial_z, (\Psi_1(x)y^i\partial_z)_{i=0}^2\}$: x, $I_{48.}$

$\mathbf{ip}_{6,D} = \{y\partial_y, \partial_x, x\partial_x, x^2\partial_x + xy\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $yz_yJ_4/J_{2,3}^2$.

Derived Series $(7, 6)$

$\mathbf{ip}_{24} = \{\partial_z, \partial_x, \partial_y + x\partial_z, x\partial_y + \frac{1}{2}x^2\partial_z, x\partial_x - y\partial_y, y\partial_x + \frac{1}{2}y^2\partial_z, x\partial_x + y\partial_y + 2z\partial_z\}$: $G_{2,1}$.

$\mathbf{ip}_{2,C}[s = n = 0, m_0 = 1] = \{\partial_x, \partial_y, 2x\partial_x, x^2\partial_x + xz\partial_z, z\partial_z, \partial_z, x\partial_z\}$: I_7.

$\mathbf{ip}_{15,C}[p = 1, \mathbf{m} = [0]] = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y + yz\partial_z, z\partial_z, (y^ie^{a_1x}\partial_z)_{i=0}^1\}$: I_{53}.

$\mathbf{ip}_{2,D}[s = 0] = \{\partial_y, \partial_x, 2x\partial_x, x^2\partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{2,2}$.

$\mathbf{ip}_{15,D} = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{1,2}$.

Derived Series $(7, 6, 5)$

$\mathbf{ip}_8[m = 0] = \{x\partial_y, x\partial_x - y\partial_y, y\partial_x, \partial_z, az\partial_z + (x\partial_x + y\partial_y), \partial_x, \partial_y\}$: $G_2/G_3^{\frac{2a-4}{3a-4}}$.

$\mathbf{ip}_{18}[h = 0] = \{x\partial_y, x\partial_x - y\partial_y, y\partial_x, \partial_z, z\partial_z, \partial_x, \partial_y\}$: $G_2/G_3^{2/3}$.

$\mathbf{ip}_{2,C}[s = 1, m_0 = n = 0] = \{\partial_x, \partial_y, x\partial_y, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y, z\partial_z, \partial_z\}$: I_{10}/z_{yy}^3.

$\mathbf{ip}_{7,C}[m = 1] = \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y, z\partial_z, \partial_z, \frac{1}{y}\partial_z, \frac{x}{y}\partial_z\}$: I_{24}^2/I_{25}.

Derived Series $(7,6,4,3)$

$\mathbf{ip}_{1,D}[s=1]=\{\partial_y, x\partial_y, \partial_x, x\partial_x+cy\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $z_y(-J_{2,1})^c J_4/J_2^2$.

$\mathbf{ip}_{3,D}[s=2]=\{\partial_y, x\partial_y, \partial_x, x\partial_x+(2y+x^2)\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{4,1}-2\log(-J_{2,1})$.

Derived Series $(7,6,4,0)$

$\mathbf{ip}^1_{1,A}[s=4,t=1]=\{\partial_x, \partial_y, x\partial_y+\partial_z, x^2\partial_y+2x\partial_z, x^3\partial_y+3x^2\partial_z,$
$x^4\partial_y+4x^3\partial_z, x\partial_x+cy\partial_y+(c-1)z\partial_z\}$: H_1, H_2.

$\mathbf{ip}^1_{1,A}[s=4,t=2]=\{\partial_x, \partial_y, x\partial_y, x^2\partial_y+\partial_z, x^3\partial_y+3x\partial_z,$
$x^4\partial_y+6x^2\partial_z, x\partial_x+cy\partial_y+(c-2)z\partial_z\}$: H_1.

$\mathbf{ip}^1_{1,A}[s=4,t=3]=\{\partial_x, \partial_y, x\partial_y, x^2\partial_y, x^3\partial_y+\partial_z, x^4\partial_y+4x\partial_z,$
$x\partial_x+cy\partial_y+(c-3)z\partial_z\}$: H_1.

$\mathbf{ip}^1_{1,A}[s=4,t=4]=\{\partial_x, \partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, x^4\partial_y+\partial_z, x\partial_x+cy\partial_y+(c-4)z\partial_z\}$:
H_1, H_8.

$\mathbf{ip}^1_{1,A}[s=4,t=5]=\{\partial_x, \partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, x^4\partial_y, x\partial_x+cy\partial_y+(c-5)z\partial_z\}$:
H_5, H_6, H_7.

$\mathbf{ip}^2_{1,A}[s=4,t=1]=\{\partial_x, \partial_y, x\partial_y+\partial_z, x^2\partial_y+2x\partial_z, x^3\partial_y+3x^2\partial_z,$
$x^4\partial_y+4x^3\partial_z, x\partial_x+y\partial_y+c\partial_z\}$: H_{11}, H_{13}.

$\mathbf{ip}^2_{1,A}[s=4,t=2]=\{\partial_x, \partial_y, x\partial_y, x^2\partial_y+\partial_z, x^3\partial_y+3x\partial_z,$
$x^4\partial_y+6x^2\partial_z, x\partial_x+2y\partial_y+c\partial_z\}$: H_{11}.

$\mathbf{ip}^2_{1,A}[s=4,t=3]=\{\partial_x, \partial_y, x\partial_y, x^2\partial_y, x^3\partial_y+\partial_z, x^4\partial_y+4x\partial_z, x\partial_x+3y\partial_y+c\partial_z\}$:
H_{11}.

$\mathbf{ip}^2_{1,A}[s=4,t>4]=\{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, x^4\partial_y, \partial_x, x\partial_x+ty\partial_y+c\partial_z\}$:
H_{15}, H_{16}, H_{17}.

$\mathbf{ip}^1_{3,A}[s=5,t=1]=\{\partial_y, x\partial_y+\partial_z, x^2\partial_y+2x\partial_z, x^3\partial_y+3x^2\partial_z, x^4\partial_y+4x^3\partial_z, \partial_x,$
$x\partial_x+(5y+x^5)\partial_y+(4z+5x^4)\partial_z\}$: H_2, H_{26}.

$\mathbf{ip}^1_{3,A}[s=5,t=2]=\{\partial_y, x\partial_y, x^2\partial_y+\partial_z, x^3\partial_y+3x\partial_z, x^4\partial_y+6x^2\partial_z, \partial_x,$

$x\partial_x + (5y + x^5)\partial_y + (3z + 10x^3)\partial_z\}$: H_{26}.

$\mathbf{ip}^1_{3,A}[s = 5, t = 3] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y + \partial_z, x^4\partial_y + 4x\partial_z, \partial_x,$
$x\partial_x + (5y + x^5)\partial_y + (2z + 10x^2)\partial_z\}$: H_{26}.

$\mathbf{ip}^1_{3,A}[s = 5, t = 4] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, x^4\partial_y + \partial_z, \partial_x,$
$x\partial_x + (5y + x^5)\partial_y + (z + 5x)\partial_z\}$: H_{26}, H_{33}.

$\mathbf{ip}^2_{3,A}[s = 5] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, x^4\partial_y, \partial_x, x\partial_x + (5y + x^5)\partial_y + c\partial_z\}$:
H_{15}, H_{16}, H_{17}.

Derived Series $(7, 6, 3)$

$\mathbf{ip}_{14,C}[m = 0, l = 3] = \{\partial_y, y\partial_y, y^2\partial_y, z\partial_z, (\Psi_i(x)\partial_z)^3_{i=1}]\}$: x, I_6.

$\mathbf{ip}_{19,D}[s = 1] = \{\partial_y, \partial_z, z\partial_z, z^2\partial_z, x\partial_y, \Psi_1(x)\partial_y, y\partial_y\}$: x, $J_{2,1}$.

$\mathbf{ip}_{20,D}[\mathbf{s} = [2]] = \{e^{a_1x}\partial_y, xe^{a_1x}\partial_y, x^2e^{a_1x}\partial_y, \partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{2,1}$.

$\mathbf{ip}_{20,D}[\mathbf{s} \in \{[1, 0], [0, 1]\}] = \{e^{a_1x}\partial_y, xe^{a_1x}\partial_y, e^{a_2x}\partial_y, \partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{2,1}$.
(or a_1 interchanged with a_2)

$\mathbf{ip}_{20,D}[\mathbf{s} = [0, 0, 0]] = \{e^{a_1x}\partial_y, e^{a_2x}\partial_y, e^{a_3x}\partial_y, \partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{2,1}$.

Derived Series $(7, 6, 0)$

$\mathbf{ip}_{19,A}[s = 4, t = 1, 2, 3] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y + [i \geq t]\varphi_i(x)\partial_z)^4_{i=1},$
$y\partial_y + z\partial_z\}_{\varphi_t=1}$: x, z_y.

$\mathbf{ip}_{19,A}[s = t = 4] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y, (\Psi_i(x)\partial_y)^3_{i=1}, \Psi_4(x)\partial_y + \partial_z\}$:
x, z_y, H_{29}.

$\mathbf{ip}_{19,A}[s = 4, t = 5] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y, (\Psi_i(x)\partial_y)^4_{i=1}\}$: x, z_y, zz_{yy}, H_{29}.

$\mathbf{ip}_{20,A}[1 + q + \sum_{i=1}^q s_i = 7] = \{\partial_x, (e^{a_ix}(x^j\partial_y + \pi_{i,j}\partial_z))^{j\to^* s_i}_{i\to q}\}$: z_y, z_{yy}.

Derived Series $(7, 5)$

$\mathbf{ip}_6[\mathbf{m} = [0]] = \{\partial_x, \partial_y, \partial_z, x\partial_y, x\partial_x + y\partial_y, x\partial_x - y\partial_y, y\partial_x\}$: G_2/G_3.

Derived Series $(7,5,3)$

$\mathbf{ip}_{2,C}[s=m_0=0,n=1]=\{\partial_x,\partial_y,2x\partial_x,x^2\partial_x,z\partial_z,y\partial_z,\partial_z\}$: I_3.

$\mathbf{ip}_{15,C}[p=0,\mathbf{m}=[1]]=\{\partial_x,\partial_y,y\partial_y,y^2\partial_y,z\partial_z,(x^ie^{a_1x}\partial_z)_{i=0}^1\}$: I_6.

$\mathbf{ip}_{15,C}[p=0,\mathbf{m}=[0,0]]=\{\partial_x,\partial_y,y\partial_y,y^2\partial_y,z\partial_z,(e^{a_ix}\partial_z)_{i=1}^2\}$: I_6.

$\mathbf{ip}_{16,C}[l=m=0]=\{\partial_x,\partial_y,x\partial_x,y\partial_y,y^2\partial_y,z\partial_z,\partial_z\}$: I_6/I_2.

$\mathbf{ip}_{4,D}[s=0]=\{\partial_x,\partial_y,y\partial_y,x\partial_x,\partial_z,z\partial_z,z^2\partial_z\}$: $J_3/(z_yJ_1)$.

$\mathbf{ip}_{21,D}[s=1,\mathbf{s}=[]]=\{\partial_y,x\partial_y,y\partial_y,\partial_x,\partial_z,z\partial_z,z^2\partial_z\}$: $J_{2,1}$.

$\mathbf{ip}_{21,D}[s=0,\mathbf{s}=[0]]=\{\partial_y,y\partial_y,\partial_x,\partial_z,z\partial_z,z^2\partial_z,e^x\partial_y\}$: $J_{2,1}$.

Derived Series $(7,5,3,0)$

$\mathbf{ip}^2_{1,A}[s=4,t=4]=\{\partial_x,\partial_y,x\partial_y,x^2\partial_y,x^3\partial_y,x^4\partial_y+\partial_z,x\partial_x+4y\partial_y+c\partial_z\}$: H_{11}, H_{19}.

$\mathbf{ip}_{4,A}[s=3,t=1]=\{\partial_y,x\partial_y+\partial_z,x^2\partial_y+2x\partial_z,x^3\partial_y+3x^2\partial_z,$
$\partial_x,y\partial_y+z\partial_z,x\partial_x-z\partial_z\}$: H_2.

$\mathbf{ip}_{4,A}[s=3,t=2]=\{\partial_y,x\partial_y,x^2\partial_y+\partial_z,x^3\partial_y+3x\partial_z,\partial_x,y\partial_y+z\partial_z,x\partial_x-2z\partial_z\}$: H_{45}.

$\mathbf{ip}_{4,A}[s=3,t=3]=\{\partial_y,x\partial_y,x^2\partial_y,x^3\partial_y+\partial_z,\partial_x,y\partial_y+z\partial_z,x\partial_x-3z\partial_z\}$: H_{19}.

$\mathbf{ip}_{4,A}[s=3,t=4]=\{\partial_y,x\partial_y,x^2\partial_y,x^3\partial_y,\partial_x,y\partial_y+z\partial_z,x\partial_x-4z\partial_z\}$: H_{19}, H_{41}.

$\mathbf{ip}_{3,C}[s=0,\mathbf{m}=[0,2]]=\{\partial_x,x\partial_x+\partial_y,z\partial_z,y\partial_z,\partial_z,x\partial_z,x^2\partial_z\}$: $(I_{15})_{i=14}$.

$\mathbf{ip}_{3,C}[s=0,\mathbf{m}=[3]]=\{\partial_x,x\partial_x+\partial_y,z\partial_z,\partial_z,x\partial_z,x^2\partial_z,x^3\partial_z\}$: $(I_{16})^{a=7}_{i=6,7}$.

$\mathbf{ip}_{10,C}[\mathbf{l}=[1,1,1,1]]=\{\partial_y,y\partial_y,z\partial_z,(y^i\Psi_1(x)\partial_z)_{i=0}^3\}$: x, I_{47}/I_{48}.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[0],[0],[0],[0]]]=\{\partial_y,x\partial_x+y\partial_y,z\partial_z,(x^{a_1}y^i\partial_z)_{i=0}^3\}$: I_{81}/I_{78}, xI_{86}/I_{78}.

Derived Series $(7,5,2,0)$

$\mathbf{ip}_{1,C}[s=0,\mathbf{m}=[2]] = \{\partial_x, \partial_y, x\partial_x + cy\partial_y, \partial_z, x\partial_z, x^2\partial_z, z\partial_z\}$: I_7/I_6^c.

$\mathbf{ip}_{1,C}[s=0,\mathbf{m}=[0,0,0]] = \{\partial_x, \partial_y, x\partial_x + cy\partial_y, \partial_z, y\partial_z, y^2\partial_z, z\partial_z\}$: I_3/I_2^c.

$\mathbf{ip}_{1,C}[s=1,\mathbf{m}=[1]] = \{\partial_x, \partial_y, x\partial_y, x\partial_x + cy\partial_y, \partial_z, x\partial_z, z\partial_z\}$: $I_{11}/(z_y^2 I_7^{2/c})$.

$\mathbf{ip}_{1,C}[s=2,\mathbf{m}=[0]] = \{\partial_x, \partial_y, x\partial_y, x^2\partial_y, x\partial_x + cy\partial_y, \partial_z, z\partial_z\}$: $I_9/(z_y^2 I_7^{1/c})$.

$\mathbf{ip}_{3,C}[s=1,\mathbf{m}=[2]] = \{\partial_x, \partial_y, x\partial_x + (y+x)\partial_y, z\partial_z, \partial_z, x\partial_z, x^2\partial_z\}$: I_{18}.

$\mathbf{ip}_{3,C}[s=2,\mathbf{m}=[1]] = \{\partial_x, \partial_y, x\partial_y, x\partial_x + (2y+x^2)\partial_y, z\partial_z, \partial_z, x\partial_z\}$: $(I_{21})_{i=11}$.

$\mathbf{ip}_{3,C}[s=3,\mathbf{m}=[0]] = \{\partial_x, \partial_y, x\partial_y, x^2\partial_y, x\partial_x + (3y+x^3)\partial_y, z\partial_z, \partial_z\}$: I_{20}.

$\mathbf{ip}_{10,C}[\mathbf{l}=[2,2]] = \{\partial_y, y\partial_y, z\partial_z, (y^i\Psi_1(x)\partial_z)_{i=0}^1, (y^i\Psi_2(x)\partial_z)_{i=0}^1\}$: x.

$\mathbf{ip}_{10,C}[\mathbf{l}=[2,1,1]] = \{\partial_y, y\partial_y, z\partial_z, (y^i\Psi_1(x)\partial_z)_{i=0}^2, \Psi_2(x)\partial_z\}$: x.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[1],[1]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, x^{a_1}\partial_z, (x^{a_1}\ln(x)^i y^j\partial_z)_{i,j=0}^1\}$:
$x^2 z_{yy}/I_{84}$.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[0,0],[0,0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_i}y^j\partial_z)_{i=1,2}^{j=0,1}\}$: $x^2 z_{yy}/I_{85}$.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[1],[0],[0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, x^{a_1}\ln(x)\partial_z, (x^{a_1}y^i\partial_z)_{i=0}^2\}$:
xI_{86}/I_{84}.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[0,0],[0],[0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}y^i\partial_z)_{i=0}^2, x^{a_2}\partial_z\}$: xI_{86}/I_{85}.

$\mathbf{ip}_{19,C}[s=0,\mathbf{j}=[0,1]] = \{\partial_y, y\partial_y, x\partial_y, z\partial_z, (v\theta_{1,1}(x)\partial_z)_{\{1,x,y\}}^{v\in}\}$: x.

Derived Series $(7,5,1,0)$

$\mathbf{ip}_{1,C}[s=0,\mathbf{m}=[0,1]] = \{\partial_x, \partial_y, x\partial_x + cy\partial_y, \partial_z, x\partial_z, y\partial_z, z\partial_z\}$: I_8.

$\mathbf{ip}_{3,C}[s=0,\mathbf{m}=[0,0,1]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, y^2\partial_z, y\partial_z, \partial_z, x\partial_z\}$: $(I_{15})_{i=14}$.

$\mathbf{ip}_{3,C}[s=1,\mathbf{m}=[0,1]] = \{\partial_x, \partial_y, x\partial_x + (y+x)\partial_y, z\partial_z, y\partial_z, \partial_z, x\partial_z\}$: $z_{yy}/I_{11}^{1/2}$.

$\mathbf{ip}_{10,C}[\mathbf{l}=[3,1]] = \{\partial_y, y\partial_y, z\partial_z, (y^i\Psi_1(x)\partial_z)_{i=0}^1, (\Psi_i(x)\partial_z)_{i=2}^3\}$: x.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[2,0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}\ln(x)\partial_z)_{i=0}^2, yx^{a_1}\partial_z\}$: xz_{yy}/I_{86}.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[1,0],[0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}\ln(x)^i y^j\partial_z)_{i+j=0}^1, x^{a_2}\partial_z\}$: xz_{yy}/I_{86}.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[0,1],[0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}y^i\partial_z)_{i=0}^1, (x^{a_2}\ln(x)^i\partial_z)_{i=0}^1\}$: xz_{yy}/I_{86}.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[0,0,0],[0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_i}\partial_z)_{i=1}^3, yx^{a_1}\partial_z\}$: xz_{yy}/I_{86}.

Derived Series $(7,5,0)$

$\mathbf{ip}_{21,A}[3 + s + h + \sum_{i=1}^h s_i = 7] = \{\partial_x, y\partial_y + z\partial_z, (x^i\partial_y + \pi_{0,i}\partial_z)_{i=0}^s,$
$(e^{a_i x}(x^j\partial_y + \pi_{i,j}\partial_z))_{i\to h}^{j\to^* s_i}\}$: z_y.

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [0,0,0,0]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, y^3\partial_z, y^2\partial_z, y\partial_z, \partial_z\}$: $(I_{16})_{i=2,3}^{a=3}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[5]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^5\}$: x, $e/e_{1,6}$, where $e \in \{e_{1,3}, e_{2,5}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[4,1]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^4, y\Psi_{1,1}(x)e^{c_1 y}\partial_z\}$: x, $(c_1 e_{1,3} - e_{3,6})/(c_1 e_{1,2} - e_{3,5})$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3,2]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^3, (y\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^2\}$: x.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3,1,1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^2, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=2}^3\}$: x.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,2,1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^2, (y^i\Psi_{1,2}(x)e^{c_1 y}\partial_z)_{i=0}^1\}$: x.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,1,1,1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^3, \Psi_{1,2}(x)e^{c_1 y}\partial_z\}$: x, $e_{1,2,4}/e_{1,2,3,5}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1,1,1,1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^4\}$: x, $e/e_{1,2}$, $e \in \{e_{1,4}, e_{3,5}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[4],[1]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^4, \Psi_{2,1}(x)e^{c_2 y}\partial_z\}$: x, $(c_2 e_{1,3} - e_{3,6})/(\Psi'_{2,1}e_{1,3} - \Psi_{2,1}e_{2,5})$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3],[2]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^3, (\Psi_{2,i}(x)e^{c_2 y}\partial_z)_{i=1}^2\}$: x.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3,1],[1]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1y}\partial_z)_{i=1}^3, (\Psi_{i,1}(x)y^je^{c_iy}\partial_z)_{j=2-i}^{i=1,2}\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,2]],[1]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,j}(x)e^{c_1y}\partial_z)_{j=1,2}^{i=0,1}, \Psi_{2,1}(x)e^{c_2y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,1],[2]]] = \{\partial_y, z\partial_z, (\Psi_{i,j}(x)e^{c_iy}\partial_z)_{i,j=1}^2, y\Psi_{1,1}(x)e^{c_1y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1],[3]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1y}\partial_z)_{i=0}^1, (\Psi_{2,i}(x)e^{c_2y}\partial_z)_{i=1}^3\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,1],[1,1]]] = \{\partial_y, z\partial_z, (y^j\Psi_{i,1}(x)e^{c_iy}\partial_z)_{i=1,2}^{j=0,1}, \Psi_{1,2}(x)e^{c_1y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,1,1]],[1]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1y}\partial_z)_{i=0}^2, (\Psi_{i,j}(x)e^{c_iy}\partial_z)_{i+j=3}\}\colon$
$x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1,1],[2]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1y}\partial_z)_{i=0}^2, (\Psi_{2,i}(x)e^{c_2y}\partial_z)_{i=1}^2\}\colon$
$x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1,1],[1,1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{i,1}(x)e^{c_iy}\partial_z)_{j=0,1}^{i=1,2}, y^2\Psi_{1,1}(x)e^{c_1y}\partial_z\}\colon$
$x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1,1,1],[1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1y}\partial_z)_{i=0}^3, \Psi_{2,1}(x)e^{c_2y}\partial_z\}\colon$
x, $(c_2e_{1,2} - e_{3,5})/(\Psi_{2,1}''e_{1,2} - \Psi_{2,1}'e_{1,4} + \Psi_{2,1}e_{2,4})$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3],[1],[1]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1y}\partial_z)_{i=1}^3, (\Psi_{i,1}(x)e^{c_iy}\partial_z)_{i=2}^3\}\colon$
$x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2],[2],[1]]] = \{\partial_y, z\partial_z, (\Psi_{i,j}(x)e^{c_iy}\partial_z)_{i=1,2}^{j=1,2}, \Psi_{3,1}(x)e^{c_3y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2,1],[1],[1]]] = \{\partial_y, z\partial_z, (\Psi_{i,j}(x)e^{c_iy}\partial_z)_{j=1,2}^{i\to j}, y\Psi_{1,1}(x)e^{c_1y}\partial_z,$
$\Psi_{3,1}(x)e^{c_3y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1],[2],[1]]] = \{\partial_y, z\partial_z, (\Psi_{i,j}(x)e^{c_iy}\partial_z)_{i=1,2}^{j\to i}, y\Psi_{1,1}(x)e^{c_1y}\partial_z,$
$\Psi_{3,1}(x)e^{c_3y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1],[1,1],[1]]] = \{\partial_y, z\partial_z, (y^j\Psi_{i,1}(x)e^{c_iy}\partial_z)_{i=1,2}^{j=1,2}, \Psi_{3,1}(x)e^{c_3y}\partial_z\}\colon$
$x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1,1],[1],[1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1y}\partial_z)_{i=0}^2, \Psi_{2,1}(x)e^{c_2y}\partial_z,$
$\Psi_{3,1}(x)e^{c_3y}\partial_z\}\colon x.$

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2],[1],[1],[1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^4, \Psi_{1,2}(x)e^{c_1 y}\partial_z\}$: x.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1,1],[1],[1],[1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^4, y\Psi_{1,1}(x)e^{c_1 y}\partial_z\}$: x.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1],[1],[1],[1],[1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^5\}$: x.

$\mathbf{ip}_{10,C}[\mathbf{l} = [4]] = \{\partial_y, y\partial_y, z\partial_z, (\Psi_i(x)\partial_z)_{i=1}^4\}$: x, I_6.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[3]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}\log(x)^i\partial_z)_{i=0}^3\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[1,1]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_i}\ln(x)^j\partial_z)_{i,j=0}^1\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[2,0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}\log(x)^i\partial_z)_{i=0}^2, x^{a_2}\partial_z\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[1,0,0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_i}\partial_z)_{i=1}^3, x^{a_1}\log(x)\partial_z\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{19,C}[s = 0, \mathbf{j} = [3]] = \{\partial_y, y\partial_y, x\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^3\}$: x.

$\mathbf{ip}_{19,C}[s = 1, \mathbf{j} = [2]] = \{\partial_y, y\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^2\}$: x.

$\mathbf{ip}_{19,C}[s = 2, \mathbf{j} = [1]] = \{\partial_y, y\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, z\partial_z, \theta_{0,1}(x)\partial_z\}$: x, I_{59}.

$\mathbf{ip}_{19,C}[s = 3, \mathbf{j} = [0]] = \{\partial_y, y\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^3, z\partial_z\}$: x, $(I_i)_{i=56}^{57}$.

Derived Series $(7,4,2,0)$

$\mathbf{ip}_{13,C}[\mathbf{J} = [[2]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (y^i e^{a_1 x}\partial_z)_{i=0}^2\}$: I_{53}.

Derived Series $(7,4,1,0)$

$\mathbf{ip}_{4,C}[s = 0, \mathbf{m} = [1]] = \{\partial_y, y\partial_y, \partial_x, x\partial_x, z\partial_z, \partial_z, x\partial_z\}$: I_{22}.

$\mathbf{ip}_{4,C}[s = 0, \mathbf{m} = [0,0]] = \{\partial_y, y\partial_y, \partial_x, x\partial_x, z\partial_z, \partial_z, y\partial_z\}$: I_{22}.

$\mathbf{ip}_{4,C}[s = 1, \mathbf{m} = [0]] = \{\partial_y, x\partial_y, y\partial_y, \partial_x, x\partial_x, z\partial_z, \partial_z\}$: $z_{yy}I_{10}/I_9^2$.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[1,0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (x^i e^{a_1 x}\partial_z)_{i=0}^1, ye^{a_1 x}\partial_z\}$: I_{76}.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[1],[0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (y^i e^{a_1 x}\partial_z)_{i=0}^1, e^{a_2 x}\partial_z\}$: I_{75}.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[0],[1]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, e^{a_1 x}\partial_z, (y^i e^{a2x}\partial_z)_{i=0}^1\}$: I_{75}.

Derived Series $(7,4,0)$

$\mathbf{ip}_{11,C}[\mathbf{L}=[[3]]]=\{\partial_x,\partial_y,z\partial_z,(y^ie^{a_1x+b_1y}\partial_z)_{i=0}^3\}$: $I/d_{1,2}$, $I\in\{d_{1,4},d_{3,5}\}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,1]]]=\{\partial_x,\partial_y,z\partial_z,(x^iy^je^{a_1x+b_1y}\partial_z)_{i,j=0}^1\}$:
$(2b_1d_{1,3}-d_{1,6})/(a_1^2d_{1,2}-d_{2,4})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[2,0]]]=\{\partial_x,\partial_y,z\partial_z,(y^ie^{a_1x+b_1y}\partial_z)_{i=0}^2,xe^{a_1x+b_1y}\partial_z\}$:
$(a_1^2d_{1,3}-d_{2,5})/(a_1^2d_{1,2}-d_{2,4})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,0,0]]]=\{\partial_x,\partial_y,z\partial_z,(x^ie^{a_1x+b_1y}\partial_z)_{i=0}^2,ye^{a_1x+b_1y}\partial_z\}$:
$(b_1^2d_{1,3}-d_{3,6})/(a_1^2d_{1,3}-d_{2,5})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0,0,0]]]=\{\partial_x,\partial_y,z\partial_z,(x^ie^{a_1x+b_1y}\partial_z)_{i=0}^2\}$: $(d_{2-i,5+i}/d_{1,3})_{i=0}^1$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1],[1]]]=\{\partial_x,\partial_y,z\partial_z,(y^ie^{a_jx+b_jy}\partial_z)_{i=0,1}^{j=1,2}\}$:
$(b_1b_2d_{1,2,6}-2d_{3,5,6})/(a_1a_2d_{1,2}-d_{2,4})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[2],[0]]]=\{\partial_x,\partial_y,z\partial_z,(y^ie^{a_1x+b_1y}\partial_z)_{i=0}^2,e^{a_2x+b_2y}\partial_z\}$:
$(b_1b_2d_{1,2}-d_{3,5})/(a_1a_2d_{1,2}-d_{2,4})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[1]]]=\{\partial_x,\partial_y,z\partial_z,(x^ie^{a_1x+b_1y}\partial_z)_{i=0}^1,(y^ie^{a_2x+b_2y}\partial_z)_{i=0}^1\}$:
$(d_{1,4,6}-2a_1d_{1,2,6}+2b_2d_{1,3,4}+4a_1b_2d_{1,2,3})/(a_1a_2d_{1,3}-d_{2,5})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[0,0]]]=\{\partial_x,\partial_y,z\partial_z,(x^ie^{a_jx+b_jy}\partial_z)_{i=0,1}^{j=1,2}\}$:
$(a_1a_2d_{1,3,4}+2d_{2,4,5})/(b_1b_2d_{1,3}-d_{3,6})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,0],[0]]]=\{\partial_x,\partial_y,z\partial_z,(ve^{a_1x+b_1y}\partial_z)_{v\in\{1,x,y\}},e^{a_2x+b_2y}\partial_z\}$:
$(d_{1,4,6}-2a_1d_{1,2,6}+2b_1d_{1,3,4}+4a_1b_1d_{1,2,3})/(a_1^2a_2d_{1,2,3}-d_{2,4,5})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0,0],[0]]]=\{\partial_x,\partial_y,z\partial_z,(x^ie^{a_1x+b_1y}\partial_z)_{i=0}^2,e^{a_2x+b_2y}\partial_z\}$:
$(b_1b_2d_{1,3}-d_{3,6})/(a_1a_2d_{1,3}-d_{2,5})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1],[0],[0]]]=\{\partial_x,\partial_y,z\partial_z,(y^ie^{a_1x+b_1y}\partial_z)_{i=0}^1,(e^{a_ix+b_iy}\partial_z)_{i=2}^3\}$:
$(d_{1,2,4,6}+2b_1d_{1,2,3,4})/\{(a_3b_2-a_2b_3)d_{1,2}+(b_3-b_2)d_{2,4}/a_1-(a_3-a_2)d_{3,5}/b_1\}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[0],[0]]]=\{\partial_x,\partial_y,z\partial_z,(x^ie^{a_1x+b_1y}\partial_z)_{i=0}^1,(e^{a_ix+b_iy}\partial_z)_{i=2}^3\}$:
$(d_{1,3,4,6}+2a_1d_{1,2,3,6})/(d_{1,3,4,5}+2a_1d_{1,2,3,5}-b_1d_{1,2,3,4})$.

$\mathbf{ip}_{11,C}[\mathbf{L} = [[0],[0],[0],[0]]] = \{\partial_x, \partial_y, z\partial_z, (e^{a_i x + b_i y}\partial_z)_{i=1}^4\}$: $d_{1,2,3,4,6}/d_{1,2,3,4,5}$.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[0,0,0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (x^i e^{a_1 x}\partial_z)_{i=0}^2\}$: I_6.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[0,0],[0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (x^i e^{a_1 x}\partial_z)_{i=0}^1, e^{a_2 x}\partial_z\}$: I_6.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[0],[0,0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, e^{a_1 x}\partial_z, (x^i e^{a_2 x}\partial_z)_{i=0}^1\}$: I_6.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[0],[0],[0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (e^{a_1 x}\partial_z)_{i=1}^3\}$: I_6.

$\mathbf{ip}_{18,C}[s = 0, \mathbf{j} = [4]] = \{\partial_y, x\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^4\}$: x, I_7.

$\mathbf{ip}_{18,C}[s = 0, \mathbf{j} = [1,1]] = \{\partial_y, x\partial_y, z\partial_z, \theta_{0,1}(x)\partial_z, (v\theta_{1,1}(x)\partial_z)_{\{1,x,y\}}^{v\in}\}$: x.

Derived Series $(7,3)$

$\mathbf{ip}_{18,D}[s = 2] = \{\partial_y, \partial_z, z\partial_z, z^2\partial_z, x\partial_y, \Psi_1(x)\partial_y, \Psi_2(x)\partial_y\}$: x, $J_{2,1}$.

Derived Series $(7,3,0)$

$\mathbf{ip}_{18,C}[s = 1, \mathbf{j} = [3]] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^2\}$: x, I_7.

Derived Series $(7,2,0)$

$\mathbf{ip}_{18,C}[s = 2, \mathbf{j} = [2]] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^2\}$: x, I_7.

Derived Series $(7,1,0)$

$\mathbf{ip}_{18,C}[s = 3, \mathbf{j} = [1]] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^3, z\partial_z, \theta_{0,1}(x)\partial_z\}$: x, I_7, I_{59}.

Derived Series $(7,0)$

$\mathbf{ip}_{18,A}[s = 5, t \in \{1,2,3,4\}] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y + [i \geq t]\varphi_i(x)\partial_z)_{i=1}^4\}_{\varphi_t = 1}$: x, z_y, z_{yy}.

$\mathbf{ip}_{18,A}[s = t = 5] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^4, \Psi_5(x) + \partial_z\}$: x, z_y, z_{yy}, H_{29}.

$\mathbf{ip}_{18,A}[s = 5, t = 6] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^5\}$: x, z, z_y, z_{yy}, H_{29}.

$\mathbf{ip}_{18,C}[s = 4, \mathbf{j} = [0]] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^4, z\partial_z\}$: x, $(I_i)_{i=63}^{65}$.

4.3.2 Groups with Six Generators

Derived series (6)

$\mathbf{p}_6 = \{\partial_x, \partial_y, \partial_z, y\partial_x - x\partial_y, z\partial_y - y\partial_z, x\partial_z - z\partial_x, x\partial_x + y\partial_y + z\partial_z\}$:
$A/B^{3/2}$, C/B^2, where

$$A := 2z_y^3 - zz_xz_y + z^2z_{xy} - 2zz_yz_{yy},\ B := z\left(z_x - z_{yy}\right),$$
$$C := 6z_y^4 + 3z^2z_x^2 - 8zz_xz_y^2 - z^3z_{xx} - 4zz_y^2z_{yy} + 4z^2z_yz_{xy}.$$

$\mathbf{p}_7 = \{\partial_y + x\partial_z, y\partial_y + z\partial_z, (xy-z)\partial_x + y^2\partial_y + yz\partial_z, \partial_x + y\partial_z, x\partial_x + z\partial_z,$
$x^2\partial_x + (xy-z)\partial_y + xz\partial_z\}$: $AC^{1/2}/B^{3/2}$, C^2D/B^2, where

$$A := -x^2z_{xx} + 4zz_{xy} - y^2z_{yy} - z_y^2z_{xx} - z_x^2z_{yy} - 2xyz_{xy} + 2xz_yz_{xx}$$
$$-2xz_xz_{xy} - 2yz_yz_{xy} + 2yz_xz_{yy} + 2z_xz_yz_{xy},$$
$$B := -z + xz_x + yz_y - z_xz_y,\ C := xy - z,\ D := z_{xy}^2 - z_{xx}z_{yy}.$$

$\mathbf{ip}_{16} = \{\partial_z, \partial_x, \partial_y + x\partial_z, x\partial_y + \frac{1}{2}x^2\partial_z, x\partial_x - y\partial_y, y\partial_x + \frac{1}{2}y^2\partial_z\}$:
$G_{3,1}$, $G_{2,1}/G_{3,1}$.

$\mathbf{ip}^1_{2,A}[s=2] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, \partial_x, 2x\partial_x + 2y\partial_y, x^2\partial_x + 2xy\partial_y + 2y\partial_z\}$:
H_{21}, H_{22}.

$\mathbf{ip}^2_{2,A}[s=2] = \{\partial_y, x\partial_y, x^2\partial_y, \partial_x, 2x\partial_x + 2y\partial_y + z\partial_z,$
$x^2\partial_x + 2xy\partial_y + z(x + cz^2)\partial_z\}$: H_{23}, H_{24}.

$\mathbf{ip}^3_{2,A} = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, \partial_x, x\partial_x + y\partial_y + c\partial_z,$
$x^2\partial_x + 2xy\partial_y + 2(cx+y)\partial_z\}$:
$2z + c\log(z_{yy}) + H_{65}/z_{yy}$, $A/z_{yy} + B$, where

$$A := 2(2z-c)z_y^2 + 4(z_xz_y + z_{xx} + 2zz_{xy}) + 2cH_{65}(2H_{33} + H_{66}),$$
$$B := 4z^2 + 8czH_{33} + (2z + 2cH_{33})H_{66} + c^2(4H_{33}^2 + H_{66}^2).$$

$\mathbf{ip}_{5,A}[s=1] = \{\partial_x, \partial_y, x\partial_y + \partial_z, x\partial_x - z\partial_z, y\partial_y + z\partial_z, x^2\partial_x + xy\partial_y + (y - xz)\partial_z\}$:
$H_{27}/(z_{yy}H_{34}^4)^{1/3}$, $H_{46}/(z_{yy}H_{34})^{2/3}$.

$\mathbf{ip}^1_{17,A} = \{\partial_x, \partial_y, x\partial_x, y\partial_y + z\partial_z, x^2\partial_x, y^2\partial_y + (2yz + cz^2)\partial_z\}$:
$H_{58}/(z_xH_{57})$, $H_{59}/H_{57}^{3/2}$.

$\mathbf{ip}^2_{17,A} = \{\partial_x, \partial_y, x\partial_x + cz\partial_z, y\partial_y + z\partial_z, x^2\partial_x + 2cxz\partial_z, y^2\partial_y + 2yz\partial_z\}$:

$H_{60}H_{62}^{(c-1)/2}$, $H_{61}H_{62}^{c}$.

$\mathbf{ip}_{6,C}[\mathbf{m}=[0]] = \{\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y + cxz\partial_z, z\partial_z, y^c\partial_z\}$: I_{29}, I_{31}.

$\mathbf{ip}_{8,C}[m=1] = \{\partial_x+\partial_y, x\partial_x+y\partial_y, x^2\partial_x+y^2\partial_y+xz\partial_z, z\partial_z, \partial_z, x\partial_z\}$:
$I_{26}I_{27,x,y,-1}^{(0,-4/3)}$, $I_{26}I_{27,y,y,-2}^{(0,-4/3)}$.

$\mathbf{ip}_{14,C}[m=1, l=1] = \{\partial_y, y\partial_y, y^2\partial_y+(y-1)z\partial_z, z\partial_z, (\Psi_1(x)y^i\partial_z)_{i=0}^{1}\}$:
x, $I_{48.}$

$\mathbf{ip}_{7,D} = \{\partial_x, 2x\partial_x+y\partial_y, x^2\partial_x+xy\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $yJ_{2,3}/z_y^2$, $y^3J_{4,1}$.

$\mathbf{ip}_{8,D} = \{\partial_x+\partial_y, x\partial_x+y\partial_y, x^2\partial_x+y^2\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$:
$\{(x-y)J_1 - 2z_xz_y\}/(z_x^{3/2}z_y^{1/2})$, $\{(x-y)J_3 - 2(z_xz_y^2+z_x^2z_y)\}/(z_xz_y)^{3/2}$.

$\mathbf{ip}_{14,D} = \{\partial_y, y\partial_y, y^2\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: x, $J_{1,2}$.

Derived series $(6,5)$

$\mathbf{ip}_3[l=1] = \{x\partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_x+y\partial_y, \partial_x, \partial_y\}$: z, G_2/G_3.

$\mathbf{ip}_5[\mathbf{m}=[0]] = \{\partial_x, \partial_y, \partial_z, x\partial_y, x\partial_x - y\partial_y, y\partial_x\}$: G_2, G_3.

$\mathbf{ip}_{20} = \{\partial_x, \partial_y, x\partial_y+\partial_z, x\partial_x - y\partial_y - 2z\partial_z, y\partial_x - z^2\partial_z, x\partial_x+y\partial_y\}$:
G_1G_4/G_5^2, $G_1^2G_3/G_5^3$.

$\mathbf{ip}_{22} = \{\partial_x, \partial_y, x\partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_x+y\partial_y+\partial_z\}$: $e^{4z}G_2$, $e^{4z}G_3$.

Derived Series $(6,5,3)$

$\mathbf{ip}_{14,C}[m=0, l=2] = \{\partial_y, y\partial_y, y^2\partial_y, z\partial_z, (\Psi_i(x)\partial_z)_{i=1}^{2}]\}$: x, I_6.

$\mathbf{ip}_{1,D}[s=0, c\neq 1] = \{\partial_x, \partial_y, x\partial_x+cy\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$:
$J_0^{c/(1-c)}J_{1,1}$, $J_0^{(2c-1)/(1-c)}J_{3,1}$.

$\mathbf{ip}_{1,D}[s=0, c=1] = \{\partial_x, \partial_y, x\partial_x+y\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: J_0, $J_3/(z_xJ_1)$.

$\mathbf{ip}_{3,D}[s=1] = \{\partial_y, \partial_x, x\partial_x+(x+y)\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{2,1}/e^{J_{0,1}}$, $J_{4,1}/e^{J_{0,1}}$.

$\mathbf{ip}_{19,D}[s=0] = \{\partial_y, \partial_z, z\partial_z, z^2\partial_z, x\partial_y, y\partial_y\}$: x, $J_{2,1}$.

$\mathbf{ip}_{20,D}[\mathbf{s}=[0,0]] = \{e^{a_1x}\partial_y, e^{a_2x}\partial_y, \partial_x, (z^i\partial_z)_{i=0}^{2}\}$: $J_{2,1}$, $a_1a_2y + \sum a_iJ_{0,1} - J_{4,1}$.

$\mathbf{ip}_{20,D}[\mathbf{s}=[1]] = \{e^{a_1x}\partial_y, xe^{a_1x}\partial_y, \partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{2,1}$, $a_1^2y + 2a_1J_{0,1} - J_{4,1}$.

Derived Series $(6, 5, 3, 0)$

$\mathbf{ip}^1_{1,A}[s=3, t=1] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, x^3\partial_y + 3x^2\partial_z,$
$\partial_x, x\partial_x + cy\partial_y + (c-1)z\partial_z\}$: H_1, H_2.

$\mathbf{ip}^1_{1,A}[s=3, t=2] = \{\partial_y, x\partial_y, x^2\partial_y + \partial_z, x^3\partial_y + 3x\partial_z, \partial_x,$
$x\partial_x + cy\partial_y + (c-2)z\partial_z\}$: H_1, $z_y^{c/2-2}H_{10}/z_{yy}$.

$\mathbf{ip}^1_{1,A}[s=3, t=3] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y + \partial_z, \partial_x, x\partial_x + cy\partial_y + (c-3)z\partial_z\}$:
H_1, H_8.

$\mathbf{ip}^1_{1,A}[s=3, t=4] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, \partial_x, x\partial_x + cy\partial_y + (c-4)z\partial_z\}$:
H_5, H_6, H_7.

$\mathbf{ip}^2_{1,A}[s=3, t=1] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, x^3\partial_y + 3x^2\partial_z,$
$\partial_x, x\partial_x + y\partial_y + c\partial_z\}$: H_{11}, H_{13}.

$\mathbf{ip}^2_{1,A}[s=3, t=2] = \{\partial_y, x\partial_y, x^2\partial_y + \partial_z, x^3\partial_y + 3x\partial_z, \partial_x, x\partial_x + 2y\partial_y + c\partial_z\}$:
H_{11}, H_{20}.

$\mathbf{ip}^2_{1,A}[s=3, t>3] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, \partial_x, x\partial_x + ty\partial_y + c\partial_z\}$: H_{15}, H_{16}, H_{17}.

$\mathbf{ip}^1_{3,A}[s=4, t=1] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, x^3\partial_y + 3x^2\partial_z, \partial_x,$
$x\partial_x + (4y + x^5)\partial_y + (3z + 4x^3)\partial_z\}$: H_2, H_{26}.

$\mathbf{ip}^1_{3,A}[s=4, t=2] = \{\partial_y, x\partial_y, x^2\partial_y + \partial_z, x^3\partial_y + 3x\partial_z, \partial_x,$
$x\partial_x + (4y + x^5)\partial_y + (2z + 6x^2)\partial_z\}$: H_{26}, H_{31}.

$\mathbf{ip}^1_{3,A}[s=4, t=3] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y + \partial_z, \partial_x,$
$x\partial_x + (4y + x^5)\partial_y + (z + 4x)\partial_z\}$: H_{26}, H_{30}.

$\mathbf{ip}^2_{3,A}[s=4] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y, \partial_x, x\partial_x + (4y + x^4)\partial_y + c\partial_z\}$:
H_{15}, H_{16}, H_{17}.

Derived Series $(6, 5, 0)$

$\mathbf{ip}_{19,A}[s = 3, t = 1, 2] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y + [i \geq t]\varphi_i(x)\partial_z)_{i=1}^3,$
$y\partial_y + z\partial_z\}_{\varphi_t=1}$: x, z_y.

$\mathbf{ip}_{19,A}[s = t = 3] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, \Psi_3(x)\partial_y + \partial_z\}$:
x, z_y, H_{29}.

$\mathbf{ip}_{19,A}[s = 3, t = 4] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^3\}$: x, z_y, zz_{yy}, H_{29}.

$\mathbf{ip}_{20,A}[1 + q + \sum_{i=1}^q s_i = 6] = \{\partial_x, (e^{a_i x}(x^j\partial_y + \pi_{i,j}\partial_z))_{i\to q}^{j\to^* s_i}\}$: z_y, z_{yy}.

Derived Series $(6, 4, 3)$

$\mathbf{ip}_{2,C}[s = n = m_0 = 0] = \{\partial_x, \partial_y, 2x\partial_x, x^2\partial_x, z\partial_z, \partial_z\}$: I_3, I_7.

$\mathbf{ip}_{15,C}[p = 0, \mathbf{m} = [0]] = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y, z\partial_z, e^{a_1 x}\partial_z\}$: I_6, I_{53}.

$\mathbf{ip}_{13,D} = \{\partial_x, \partial_y, y\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{1,2}$, $J_{3,2}$.

$\mathbf{ip}_{21,D}[s = 0, \mathbf{s} = []] = \{\partial_y, y\partial_y, \partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $J_{1,2}$, $J_{3,2}$.

Derived Series $(6, 4, 2, 0)$

$\mathbf{ip}^2_{1,A}[s = 3, t = 3] = \{\partial_y, x\partial_y, x^2\partial_y, x^3\partial_y + \partial_z, \partial_x, x\partial_x + 3y\partial_y + c\partial_z\}$: H_{11}, H_{19}.

$\mathbf{ip}_{4,A}[s = 2, t = 1] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, \partial_x, y\partial_y + z\partial_z, x\partial_x - z\partial_z\}$:
H_2, $H_{43} + H_{44}$.

$\mathbf{ip}_{4,A}[s = 2, t = 2] = \{\partial_y, x\partial_y, x^2\partial_y + \partial_z, \partial_x, y\partial_y + z\partial_z, x\partial_x - 2z\partial_z\}$:
H_{19}, $H_{42} + 2H_{41}$.

$\mathbf{ip}_{4,A}[s = 2, t = 3] = \{\partial_y, x\partial_y, x^2\partial_y, \partial_x, y\partial_y + z\partial_z, x\partial_x - 3z\partial_z\}$: H_{19}, H_{41}.

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [2]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, \partial_z, x\partial_z, x^2\partial_z\}$: $(I_{16})_{i=6,7}^{a=7}$.

$\mathbf{ip}_{10,C}[\mathbf{l} = [1, 1, 1]] = \{\partial_y, y\partial_y, z\partial_z, (y^i\Psi_1(x)\partial_z)_{i=0}^2\}$: x, I_{47}/I_{48}.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[0], [0], [0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}y^i\partial_z)_{i=0}^2\}$: xI_{86}/I_{78}, I_{81}/I_{78}.

Derived Series $(6, 4, 1, 0)$

$\mathbf{ip}_{1,C}[s = 0, \mathbf{m} = [1]] = \{\partial_x, \partial_y, x\partial_x + cy\partial_y, \partial_z, x\partial_z, z\partial_z\}$: $I_5^{1/(c-2)} I_6$, $I_5^{c/(c-2)} I_7$.

$\mathbf{ip}_{1,C}[s = 0, \mathbf{m} = [0, 0]] = \{\partial_x, \partial_y, x\partial_x + cy\partial_y, \partial_z, y\partial_z, z\partial_z\}$: I_3/I_2^c, I_4/I_2^{2c-1}.

$\mathbf{ip}_{1,C}[s = 1, \mathbf{m} = [0]] = \{\partial_x, \partial_y, x\partial_y, x\partial_x + cy\partial_y, \partial_z, z\partial_z\}$:
$I_9/(z_y^2 I_7^{1/c})$, $I_7^{1-2/c} I_{10}/z_y^3$.

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [0, 1]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, y\partial_z, \partial_z, x\partial_z\}$: $(I_{16})_{i=12,13}^{a=11}$.

$\mathbf{ip}_{3,C}[s = 1, \mathbf{m} = [1]] = \{\partial_x, \partial_y, x\partial_x + (y + x)\partial_y, z\partial_z, \partial_z, x\partial_z\}$: I_{18}, I_{19}.

$\mathbf{ip}_{3,C}[s = 2, \mathbf{m} = [0]] = \{\partial_x, \partial_y, x\partial_y, x\partial_x + (2y + x^2)\partial_y, z\partial_z, \partial_z\}$: I_{20}, $(I_{21})_{i=10}$.

$\mathbf{ip}_{10,C}[\mathbf{l} = [2, 1]] = \{\partial_y, y\partial_y, z\partial_z, (y^i \Psi_1(x)\partial_z)_{i=0}^1, \Psi_2(x)\partial_z\}$: x, $z_{yy} I_{49}/I_{50}^2$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[1], [0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1} \ln(x)^i y^j \partial_z)_{i+j=0}^1\}$:
xI_{86}/I_{84}, $x^2 z_{yy}/I_{84}$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[0, 0], [0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1} y^i \partial_z)_{i=0}^1, x^{a_2}\partial_z\}$:
xI_{86}/I_{85}, $x^2 z_{yy}/I_{85}$.

Derived Series $(6, 4, 0)$

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [0, 0, 0]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, y^2\partial_z, y\partial_z, \partial_z\}$: $(I_{16})_{i=2,3}^{a=3}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[4]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x) e^{c_1 y}\partial_z)_{i=1}^4\}$: x, $e/e_{1,6}$, where $e \in \{e_{1,3}, e_{2,5}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3, 1]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x) e^{c_1 y}\partial_z)_{i=1}^3, y\Psi_{1,1}(x) e^{c_1 y}\partial_z\}$:
x, $e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2, 2]]] = \{\partial_y, z\partial_z, (y^i \Psi_{1,j}(x) e^{c_1 y}\partial_z)_{j=1,2}^{i=0,1}\}$: x, $e_{1,2,3,4}/e_{1,2,3,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2, 1, 1]]] = \{\partial_y, z\partial_z, (y^i \Psi_{1,1}(x) e^{c_1 y}\partial_z)_{i=0}^2, \Psi_{1,2}(x) e^{c_1 y}\partial_z\}$:
x, $e_{1,2,3,4}/e_{1,2,3,5}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1, 1, 1, 1]]] = \{\partial_y, z\partial_z, (y^i \Psi_{1,1}(x) e^{c_1 y}\partial_z)_{i=0}^3\}$:
x, $e/e_{1,2}$, $e \in \{e_{1,4}, e_{3,5}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[3],[1]]]=\{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^3, \Psi_{2,1}(x)e^{c_2 y}\partial_z\}$:
$x,\ e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[2],[2]]]=\{\partial_y, z\partial_z, (\Psi_{i,j}(x)e^{c_i y}\partial_z)_{i=1,2}^{j=1,2}\}$:
$x,\ e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[2,1],[1]]]=\{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^2, (\Psi_{i,1}(x)y^j e^{c_i y}\partial_z)_{j=2-i}^{i=1,2}\}$:
$x,\ e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[1,1],[2]]]=\{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^1, (\Psi_{2,i}(x)e^{c_2 y}\partial_z)_{i=1}^2\}$:
$x,\ e_{1,2,3,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[1,1,1],[1]]]=\{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^2, \Psi_{2,1}(x)e^{c_2 y}\partial_z\}$:
$x,\ e_{1,2,3,4,6}/e_{1,2,3,5,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[1,1],[1,1]]]=\{\partial_y, z\partial_z, (y^j\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1,2}^{j=1,2}\}$:
$x,\ e_{1,2,3,4,5}/e_{1,2,3,5,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[2],[1],[1]]]=\{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^3, \Psi_{1,2}(x)e^{c_1 y}\partial_z\}$:
$x,\ e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[1,1],[1],[1]]]=\{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i\to 3}, y\Psi_{1,1}(x)e^{c_1 y}\partial_z\}$:
$x,\ e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[1],[1],[1],[1]]]=\{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^4\}$: $x,\ e_{1,2,3,4,5}/e_{1,2,3,4,6}$.

$\mathbf{ip}_{10,C}[\mathbf{l}=[3]]=\{\partial_y, y\partial_y, z\partial_z, (\Psi_i(x)\partial_z)_{i=1}^3\}$: $x,\ I_6$.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[2]]]=\{\partial_y, x\partial_x+y\partial_y, z\partial_z, (x^{a_1}\log(x)^i\partial_z)_{i=0}^2\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[1,0]]]=\{\partial_y, x\partial_x+y\partial_y, z\partial_z, (x^{a_1}\log(x)^i\partial_z)_{i=0}^1, x^{a_2}\partial_z\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[0,0,0]]]=\{\partial_y, x\partial_x+y\partial_y, z\partial_z, (x^{a_i}\partial_z)_{i=1}^3\}$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{19,C}[s=0,\mathbf{j}=[2]]=\{\partial_y, y\partial_y, x\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^2\}$: $x,\ I_{61}/z_y^2$.

$\mathbf{ip}_{19,C}[s=1,\mathbf{j}=[1]]=\{\partial_y, y\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z, \theta_{0,1}(x)\partial_z\}$: $x,\ I_{59}$.

Derived Series $(6,3)$

$\mathbf{ip}_{18,D}[s=1] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: x, $J_{2,1}$.

Derived Series $(6,3,1,0)$

$\mathbf{ip}_{13,C}[\mathbf{J}=[[1]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (y^i e^{a_1 x}\partial_z)_{i=0}^1\}$: I_{53}, I_{72}.

Derived Series $(6,3,0)$

$\mathbf{ip}_{4,C}[s=0,\mathbf{m}=[0]] = \{\partial_y, y\partial_y, \partial_x, x\partial_x, z\partial_z, \partial_z\}$: $I_1 I_{12}$, $I_1^2 I_{13}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[2]]] = \{\partial_x, \partial_y, z\partial_z, (y^i e^{a_1 x + b_1 y}\partial_z)_{i=0}^2\}$: $I/d_{1,2}$, $I \in \{d_{1,4}, d_{3,5}\}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,0]]] = \{\partial_x, \partial_y, z\partial_z, (v e^{a_1 x + b_1 y}\partial_z)_{v\in\{1,x,y\}}\}$:
$(b_1^2 d_{1,i} - d_{3,i+3})/(a_1^2 d_{1,2} - d_{2,4})_{i=2}^3$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0,0]]] = \{\partial_x, \partial_y, z\partial_z, (x^i e^{a_1 x + b_1 y}\partial_z)_{i=0}^2\}$: $(d_{2-i,5+i}/d_{1,3})_{i=0}^1$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1],[0]]] = \{\partial_x, \partial_y, z\partial_z, (y^i e^{a_1 x + b_1 y}\partial_z)_{i=0}^1, e^{a_2 x + b_2 y}\partial_z\}$:
$I/d_{1,2,4}$, $I \in \{b_1 b_2 d_{1,2} - d_{3,5}, d_{1,2,6} - 2b_1 d_{1,2,3}\}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[0]]] = \{\partial_x, \partial_y, z\partial_z, (x^i e^{a_1 x + b_1 y}\partial_z)_{i=0}^1, e^{a_2 x + b_2 y}\partial_z\}$:
$I/(2a_1 d_{1,2,3} + d_{1,3,4})$, $I \in \{a_1 a_2 d_{1,3} - d_{2,5}, b_1 b_2 d_{1,3} - d_{3,6}\}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0],[0],[0]]] = \{\partial_x, \partial_y, z\partial_z, (e^{a_i x + b_i y}\partial_z)_{i=1}^3\}$: $(d_{1,2,3,i}/d_{1,2,3,4})_{i=5}^6$.

$\mathbf{ip}_{13,C}[\mathbf{J}=[[0,0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (x^i e^{a_1 x}\partial_z)_{i=0}^1\}$: I_6, I_{71}.

$\mathbf{ip}_{13,C}[\mathbf{J}=[[0],[0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, (e^{a_i x}\partial_z)_{i=1}^2\}$: I_6, I_{74}.

$\mathbf{ip}_{18,C}[s=0,\mathbf{j}=[3]] = \{\partial_y, x\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^3\}$: x, I_7.

$\mathbf{ip}_{18,C}[s=0,\mathbf{j}=[0,1]] = \{\partial_y, x\partial_y, z\partial_z, (v\theta_{1,1}(x)\partial_z)_{\{1,x,y\}}^{v\in}\}$: x, I_{67}/z_{yy}^2.

$\mathbf{ip}_{19,C}[s=2,\mathbf{j}=[0]] = \{\partial_y, y\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, z\partial_z\}$: x, $(I_i)_{i=56}^{57}$.

Derived Series $(6,2,0)$

$\mathbf{ip}_{18,C}[s=1,\mathbf{j}=[2]] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^2\}$: x, I_7.

Derived Series $(6,1,0)$

$\mathbf{ip}_{18,C}[s=2,\mathbf{j}=[1]] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, z\partial_z, \theta_{0,1}(x)\partial_z\}$: x, I_7, I_{59}.

Derived Series $(6,0)$

$\mathbf{ip}_{18,A}[s=4, t\in\{1,2,3\}] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y + [i\geq t]\varphi_i(x)\partial_z)_{i=1}^4\}_{\varphi_t=1}$:
x, z_y, z_{yy}.

$\mathbf{ip}_{18,A}[s=t=4] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^3, \Psi_4(x)+\partial_z\}$: x, z_y, z_{yy}, H_{29}.

$\mathbf{ip}_{18,A}[s=4,t=5] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^4\}$: x, z, z_y, z_{yy}, H_{29}.

$\mathbf{ip}_{18,C}[s=3,\mathbf{j}=[0]] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^3, z\partial_z\}$: x, $(I_i)_{i=63}^{65}$.

4.3.3 Groups with Five Generators

Derived Series (5)

$\mathbf{ip}_2[l=1] = \{x\partial_y, x\partial_x - y\partial_y, y\partial_x, \partial_x, \partial_y\}$: z, G_2, G_3.

$\mathbf{ip}_{13} = \{\partial_x, \partial_y, x\partial_y+\partial_z, x\partial_x - y\partial_y - 2z\partial_z, y\partial_x - z^2\partial_z\}$:
G_3/G_1^2, $G_4/G_1^{5/3}$, $G_5/G_1^{4/3}$.

$\mathbf{ip}^1_{2,A}[s=1] = \{\partial_y, x\partial_y+\partial_z, \partial_x, 2x\partial_x+y\partial_y-z\partial_z, x^2\partial_x+xy\partial_y+(-xz+y)\partial_z\}$:
H_{21}, H_{22}, H_{34}/z_{yy}.

$\mathbf{ip}^2_{2,A}[s=1] = \{\partial_y, x\partial_y, \partial_x, 2x\partial_x+y\partial_y+z\partial_z, x^2\partial_x+xy\partial_y+z(x+cz^2)\partial_z\}$:
H_{23}, H_{24}, H_{25}.

Derived Series $(5,4)$

$\mathbf{ip}_{15,C}[p>0,\mathbf{m}=[]] = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y+pyz\partial_z, z\partial_z\}$: z_x/z, z_{xx}/z, I_{52}.

Derived Series $(5,4,3)$

$\mathbf{ip}^1_{16,A} = \{\partial_x, \partial_y, x\partial_x, y\partial_y+z\partial_z, y^2\partial_y+(2yz+cz^2)\partial_z\}$:
$H_{56}H_{57}^{1/2}/z_x^2$, $H_{58}/(z_xH_{57})^{1/2}$, $H_{59}/H_{57}^{3/2}$.

$\mathbf{ip}^2_{16,A} = \{\partial_x, \partial_y, x\partial_x+cz\partial_z, y\partial_y+z\partial_z, y^2\partial_y+2yz\partial_z\}$: $zz_x^{-2}z_{xx}$, H_{60}, H_{61}.

$\mathbf{ip}^3_{16,A} = \{\partial_x, \partial_y, x\partial_x + z\partial_z, y\partial_y, y^2\partial_y\}$: z_x, zz_{xx}, zz_{xy}/z_y.

$\mathbf{ip}_{7,C}[m=0] = \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y, z\partial_z, \partial_z\}$: yI_7, y^3I_{10}/z_y^3, yI_{23}/z_y^2.

$\mathbf{ip}_{8,C}[m=0] = \{\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y, z\partial_z, \partial_z\}$:
$I^{(-3/2,1/2)}_{27,x,x,2}$, $I^{(-1/2,-1/2)}_{27,x,y,0}$, $I^{(1/2,-3/2)}_{27,y,y,-2}$.

$\mathbf{ip}_{14,C}[m=0, l=1] = \{\partial_y, y\partial_y, y^2\partial_y, z\partial_z, \Psi_1(x)\partial_z\}$: x, I_6, I_{48}.

$\mathbf{ip}_{3,D}[s=0] = \{\partial_y, \partial_x, x\partial_x + (x+y)\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: J_0/e^y, $J_{1,1}$, $J_{3,1}/e^y$.

$\mathbf{ip}_{10,D} = \{\partial_y, y\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: x, $J_{1,2}$, $J_{3,2}$.

$\mathbf{ip}_{12,D} = \{\partial_y, x\partial_x + y\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: J_0, $xJ_{1,1}$, $xJ_{3,1}$.

$\mathbf{ip}_{20,D}[\mathbf{s}=[0]] = \{e^{a_1x}\partial_y, \partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: $a_1y + J_{0,1}$, $J_{2,1}$, $a_1^2y + 2a_1J_{0,1} - J_{4,1}$.

Derived Series $(5,4,2,0)$

$\mathbf{ip}^2_{1,A}[s=2, t=1] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, \partial_x, x\partial_x + y\partial_y + c\partial_z\}$:
H_{11}, H_{13}, $H_{12} + H_{14}$.

$\mathbf{ip}^1_{1,A}[s=2, t=2] = \{\partial_y, x\partial_y, x^2\partial_y + \partial_z, \partial_x, x\partial_x + cy\partial_y + (c-2)z\partial_z\}$:
H_1, H_8, $z_y^{c/2-4}(2z_y^3z + H_9)$.

$\mathbf{ip}^1_{1,A}[s=2, t=3] = \{\partial_y, x\partial_y, x^2\partial_y, \partial_x, x\partial_x + cy\partial_y + (c-3)z\partial_z\}$: H_5, H_6, H_7.

$\mathbf{ip}^2_{1,A}[s=2, t=1] = \{\partial_y, x\partial_y + \partial_z, x^2\partial_y + 2x\partial_z, \partial_x, x\partial_x + y\partial_y + c\partial_z\}$:
H_{11}, H_{13}, $H_{12} + H_{14}$.

$\mathbf{ip}^2_{1,A}[s=2, t>2] = \{\partial_y, x\partial_y, x^2\partial_y, \partial_x, x\partial_x + ty\partial_y + c\partial_z\}$: H_{15}, H_{16}, H_{17}.

$\mathbf{ip}^1_{3,A}[s=3, t=1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + (3y + x^3)\partial_y + (2z + 3x^2)\partial_z,$
$x^2\partial_y + 2x\partial_z\}$: H_2, H_{26}, $z_xz_y + zz_y^2 + H_{18} + 6H_{33}$.

$\mathbf{ip}^1_{3,A}[s=3, t=2] = \{\partial_y, x\partial_y, x^2\partial_y + \partial_z, \partial_x, x\partial_x + (3y + x^3)\partial_y + (z + 3x)\partial_z\}$:
H_{26}, H_{30}, H_{32}.

$\mathbf{ip}^2_{3,A}[s=3] = \{\partial_y, x\partial_y, x^2\partial_y, \partial_x, x\partial_x + (3y + x^3)\partial_y + c\partial_z\}$: H_{15}, H_{16}, H_{17}.

Derived Series $(5,4,0)$

$\mathbf{ip}_{19,A}[s=2,t=1]=\{\partial_y, y\partial_y+z\partial_z, x\partial_y, \Psi_1(x)\partial_y+\partial_z, \Psi_2(x)\partial_y+\varphi_2(x)\partial_z\}$:
$x,\ z_y,\ H_{63}$.

$\mathbf{ip}_{19,A}[s=2,t=2]=\{\partial_y, y\partial_y+z\partial_z, x\partial_y, \Psi_1(x)\partial_y, \Psi_2(x)\partial_y+\partial_z\}$:
$x,\ z_y,\ H_{29}$.

$\mathbf{ip}_{19,A}[s=2,t=3]=\{\partial_y, y\partial_y+z\partial_z, x\partial_y, \Psi_1(x)\partial_y, \Psi_2(x)\partial_y\}$:
$x,\ z_y,\ zz_{yy},\ H_{29}$.

Derived Series $(5,3)$

$\mathbf{ip}_{15,C}[p=0,\mathbf{m}=[]]=\{\partial_x, \partial_y, y\partial_y, y^2\partial_y, z\partial_z\}$: $z_x/z,\ z_{xx}/z,\ I_6$.

$\mathbf{ip}_{11,D}=\{\partial_x, \partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $J_0,\ J_{1,1},\ J_{3,1}$.

$\mathbf{ip}_{18,D}[s=0]=\{\partial_y, x\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: $x,\ J_{2,1},\ J_{4,1}$.

Derived Series $(5,3,1,0)$

$\mathbf{ip}^2_{1,A}[s=2,t=2]=\{\partial_y, x\partial_y, x^2\partial_y+\partial_z, \partial_x, x\partial_x+2y\partial_y+c\partial_z\}$:
$H_{11},\ H_{19},\ 2z+H_9/z_y^3+cH_{33}$.

$\mathbf{ip}_{4,A}[s=1,t=1]=\{\partial_y, x\partial_y+\partial_z, \partial_x, y\partial_y+z\partial_z, x\partial_x-z\partial_z\}$:
$H_2,\ H_{43},\ H_{44}$.

$\mathbf{ip}_{4,A}[s=1,t=2]=\{\partial_y, x\partial_y, \partial_x, y\partial_y+z\partial_z, x\partial_x-2z\partial_z\}$: $H_{19},\ H_{41},\ H_{42}$.

$\mathbf{ip}_{3,C}[s=0,\mathbf{m}=[1]]=\{\partial_x, x\partial_x+\partial_y, z\partial_z, \partial_z, x\partial_z\}$: $(I_{16})^{a=7}_{i=5,6,7}$.

$\mathbf{ip}_{10,C}[\mathbf{l}=[1,1]]=\{\partial_y, y\partial_y, z\partial_z, (y^i\Psi_1(x)\partial_z)^1_{i=0}\}$: $x,\ I_{47}/I_{48},\ z_{yy}I_{47}/I^2_{50}$.

$\mathbf{ip}_{12,C}[\mathbf{U}=[[0],[0]]]=\{\partial_y, x\partial_x+y\partial_y, z\partial_z, (x^{a_1}y^i\partial_z)^1_{i=0}\}$: $I_{80},\ xI_{86}/I_{78},\ I_{81}/I_{78}$.

Derived Series $(5, 3, 0)$

$\mathbf{ip}_{1,C}[s = 0, \mathbf{m} = [0]] = \{\partial_x, \partial_y, x\partial_x + cy\partial_y, \partial_z, z\partial_z\}$:
$I_1^{1/(c-1)} I_2,\ I_1^{c/(c-1)} I_3,\ I_1^{(2c-1)/(c-1)} I_4$.

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [0, 0]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, y\partial_z, \partial_z\}$: $(I_{16})_{i=2,3,4}^{a=3}$.

$\mathbf{ip}_{3,C}[s = 1, \mathbf{m} = [0]] = \{\partial_x, \partial_y, x\partial_x + (y + x)\partial_y, z\partial_z, \partial_z\}$: $(I_{17})_{i=7,9,10}^{a=7}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[3]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)_{i=1}^3\}$: x, $e/e_{1,6}$, where $e \in \{e_{1,3}, e_{2,5}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2, 1]]] = \{\partial_y, z\partial_z, (y^i \Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^1, \Psi_{1,2}(x)e^{c_1 y}\partial_z\}$:
x, $(e_{1,2,3,i}/e_{1,2,4})_{i=5}^6$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1, 1, 1]]] = \{\partial_y, z\partial_z, (y^i \Psi_{1,1}(x)e^{c_1 y}\partial_z)_{i=0}^2\}$:
x, $e/e_{1,2}$, where $e \in \{e_{1,4}, e_{3,5}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2], [1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^2, \Psi_{1,2}(x)e^{c_1 y}\partial_z\}$:
x, $(e_{1,2,3,i}/e_{1,2,3,4})_{i=5}^6$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1, 1], [1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^2, \Psi_{1,1}(x)ye^{c_1 y}\partial_z\}$:
x, $(e_{1,2,3,i}/e_{1,2,3,4})_{i=5}^6$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1], [1], [1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)_{i=1}^3\}$: x, $(e_{1,2,3,i}/e_{1,2,3,4})_{i=5,6}$.

$\mathbf{ip}_{10,C}[\mathbf{l} = [2]] = \{\partial_y, y\partial_y, z\partial_z, (\Psi_i(x)\partial_z)_{i=1}^2\}$: x, I_6, $I_{46}I_{49}$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[1]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_1}\log(x)^i\partial_z)_{i=0}^1\}$: $(I_i)_{i=82}^{83}$, $I_{84}/(xz_y)$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[0, 0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, (x^{a_i}\partial_z)_{i=1}^2\}$: $(I_i)_{i=82}^{83}$, $I_{85}/(xz_y)$.

$\mathbf{ip}_{19,C}[s = 0, \mathbf{j} = [1]] = \{\partial_y, y\partial_y, x\partial_y, z\partial_z, \theta_{0,1}(x)\partial_z\}$: x, I_{59}, $z_{yy}I_{60}/z_y^4$.

$\mathbf{ip}_{19,C}[s = 1, \mathbf{j} = [0]] = \{\partial_y, y\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z\}$: x, $(I_i)_{i=56}^{57}$.

Derived Series $(5, 2, 0)$

$\mathbf{ip}_{11,C}[\mathbf{L} = [[0, 0]]] = \{\partial_x, \partial_y, z\partial_z, (x^i e^{a_1 x + b_1 y}\partial_z)_{i=0}^1\}$:
$I/d_{1,3}$, $I \in \{d_{1,6}, d_{2,5}, a_1^2 d_{1,2} - d_{2,4}\}$.

$\mathbf{ip}_{11,C}[\mathbf{L} = [[0], [0]]] = \{\partial_x, \partial_y, z\partial_z, (e^{a_i x + b_i y}\partial_z)_{i=1}^2\}$: $(d_{1,2,i}/d_{1,2,3})_{i=4}^6$.

$\mathbf{ip}_{11,C}[\mathbf{L} = [[1]]] = \{\partial_x, \partial_y, z\partial_z, (y^i e^{a_1 x + b_1 y}\partial_z)_{i=0}^1\}$:
$I/d_{1,2}$, $I \in \{d_{1,4}, d_{3,5}, b_1^2 d_{1,3} - d_{3,6}\}$.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[0]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z, e^{a_1 x}\partial_z\}$: I_6, I_{53}, I_{70}.

$\mathbf{ip}_{18,C}[s = 0, \mathbf{j} = [2]] = \{\partial_y, x\partial_y, z\partial_z, (\theta_{0,i}(x)\partial_z)_{i=1}^2\}$: x, I_7, $I_{61}/(z_y z_{yy})$.

Derived Series $(5, 1, 0)$

$\mathbf{ip}_{18,C}[s = 1, \mathbf{j} = [1]] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z, \theta_{0,1}(x)\partial_z\}$: x, I_7, I_{59}.

Derived Series $(5, 0)$

$\mathbf{ip}_{18,C}[s = 2, \mathbf{j} = [0]] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, z\partial_z\}$: x, $(I_i)_{i=63}^{65}$.

$\mathbf{ip}_{18,A}[s = 3, t \in \{1, 2\}] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y + [i \geq t]\varphi_i(x)\partial_z)_{i=1}^3\}_{\varphi_t = 1}$:
x, z_y, z_{yy}.

$\mathbf{ip}_{18,A}[s = t = 3] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^2, \Psi_3(x) + \partial_z\}$: x, z_y, z_{yy}, H_{29}.

$\mathbf{ip}_{18,A}[s = 3, t = 4] = \{\partial_y, x\partial_y, (\Psi_i(x)\partial_y)_{i=1}^3\}$: x, z, z_y, z_{yy}, H_{29}.

4.3.4 Groups with Four Generators

Derived Series (4)

$\mathbf{ip}_{6,A}[a = 0] = \{\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y\}$: z, H_{49}, H_{50}, $H_9/(H_{49}H_{52})^2$.

$\mathbf{ip}_{6,A}[a = 1] = \{\partial_x, y\partial_y, x\partial_x - z\partial_z, x^2\partial_x + xy\partial_y - 2xz\partial_z\}$:
H_{49}/z, H_{50}/z, $y(2zz_{xy} + H_{52})/(z^2 H_{53})$, $(y^2 H_9 + H_{54})/(z^3 H_{53}^2)$.

$\mathbf{ip}_{14,C}[m > 0, l = 0] = \{\partial_y, y\partial_y, y^2\partial_y + m(y-1)z\partial_z, z\partial_z\}$: x, z_x/z, z_{xx}/z, I_{52}.

Derived Series $(4,3)$

$\mathbf{ip}^1_{2,A}[s=0] = \{\partial_y, \partial_x, 2x\partial_x - 2z\partial_z, x^2\partial_x - 2xz\partial_z\}$:
z_y/z, z_{yy}/z, $(zz_{xx} - z_x z_y)/z^3$, $(2zz_{xx} - 3z_x^2)/z^4$.

$\mathbf{ip}^1_{15,A} = \{\partial_x, \partial_y, y\partial_y + z\partial_z, y^2\partial_y + (2yz + cz^2)\partial_z\}$:
$H_{56}/(zz_x)$, $(z/z_x)^2 H_{57}$, zH_{58}/z_x^2, $(z/z_x)^3 H_{59}$.

$\mathbf{ip}^2_{15,A} = \{\partial_x, \partial_y, y\partial_y, y^2\partial_y\}$: z, z_x, z_{xx}, z_y/z_{xy}.

$\mathbf{ip}_{14,C}[m=l=0] = \{\partial_y, y\partial_y, y^2\partial_y, z\partial_z\}$: x, z_x/z, z_{xx}/z, I_6.

$\mathbf{ip}_{9,D} = \{\partial_y, \partial_z, z\partial_z, z^2\partial_z\}$: x, J_0, $J_{1,1}$, $J_{3,1}$.

$\mathbf{ip}_{20,D}[\mathbf{s}=[]] = \{\partial_x, \partial_z, z\partial_z, z^2\partial_z\}$: x, J_0, $J_{1,1}$, $J_{3,1}$.

Derived Series $(4,3,1,0)$

$\mathbf{ip}^1_{1,A}[s=1,t=1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + cy\partial_y + (c-1)z\partial_z\}$:
H_1, H_2, H_3, H_4.

$\mathbf{ip}^1_{1,A}[s=1,t=2] = \{\partial_y, x\partial_y, \partial_x, x\partial_x + cy\partial_y + (c-2)z\partial_z\}$:
H_5, H_6, H_7, $z^{(4-c)/(c-2)}H_9/z_y^2$.

$\mathbf{ip}^2_{1,A}[s=1,t>1] = \{\partial_y, x\partial_y, \partial_x, x\partial_x + ty\partial_y + c\partial_z\}$:
H_{15}, H_{16}, H_{17}, $H_9(H_{51})_{a=2}/z_y^2$.

$\mathbf{ip}^1_{3,A}[s=2,t=1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + (2y + x^2)\partial_y + (z + 2x)\partial_z\}$:
H_2, H_{26}, H_{28}, $2H_{33} - H_{18}/z_y$.

$\mathbf{ip}^2_{3,A}[s=2] = \{\partial_y, x\partial_y, \partial_x, x\partial_x + (2y + x^2)\partial_y + c\partial_z\}$: H_{15}, H_{16}, H_{17}, H_{38}.

Derived Series $(4,3,0)$

$\mathbf{ip}_{19,A}[s=t=1] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y, \Psi_1(x)\partial_y + \partial_z\}$: x, z_y, H_{29}, $z_{yy}H_{64}$.

$\mathbf{ip}_{19,A}[s=1,t=2] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y, \Psi_1(x)\partial_y\}$: x, z_y, zz_{yy}, H_{29}.

Derived Series $(4, 2, 0)$

$\mathbf{ip}_{4,A}[s = 0, t = 1] = \{\partial_y, \partial_x, y\partial_y + z\partial_z, x\partial_x - z\partial_z\}$:
$zz_y/z_x,\ zz_{xx}/z_x^2,\ z^2 z_{xy}/z_x^2,\ z^3 z_{yy}/z_x^2$.

$\mathbf{ip}^2_{1,A}[s = 1, t = 1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + y\partial_y + c\partial_z\}$: $H_{11},\ H_{12},\ H_{13},\ H_{14}$.

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [0]] = \{\partial_x, x\partial_x + \partial_y, z\partial_z, \partial_z\}$: $(I_{15})_{i=1},\ (I_{16})^{a=3}_{i=2,3,4}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[2]]] = \{\partial_y, z\partial_z, (\Psi_{1,i}(x)e^{c_1 y}\partial_z)^2_{i=1}\}$:
$x,\ e/e_{1,3}$, where $e \in \{e_{1,6},\ e_{2,5},\ e_{1,3,6}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1, 1]]] = \{\partial_y, z\partial_z, (y^i\Psi_{1,1}(x)e^{c_1 y}\partial_z)^1_{i=0}\}$:
$x,\ e/e_{1,2}$, where $e \in \{e_{1,4},\ e_{3,5},\ e_{1,2,3}\}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1], [1]]] = \{\partial_y, z\partial_z, (\Psi_{i,1}(x)e^{c_i y}\partial_z)^2_{i=1}\}$: $x,\ (e_{1,2,i}/e_{1,2,3})_{i=4,5,6}$.

$\mathbf{ip}_{10,C}[\mathbf{l} = [1]] = \{\partial_y, y\partial_y, z\partial_z, \Psi_1(x)\partial_z\}$: $x,\ I_6,\ I_{46}I_{47},\ I_{47}/I_{48}$.

$\mathbf{ip}_{12,C}[\mathbf{U} = [[0]]] = \{\partial_y, x\partial_x + y\partial_y, z\partial_z, x^{a_1}\partial_z\}$: $xz_y/I_{78},\ I_{80},\ x^2 z_{xy}/I_{78},\ I_{81}/I_{78}$.

$\mathbf{ip}_{19,C}[s = 0, \mathbf{j} = [0]] = \{\partial_y, y\partial_y, x\partial_y, z\partial_z\}$: $x,\ (I_i)^{58}_{i=56}$.

Derived Series $(4, 1, 0)$

$\mathbf{ip}_{11,C}[\mathbf{L} = [[0]]] = \{\partial_x, \partial_y, z\partial_z, e^{a_1 x + b_1 y}\partial_z\}$: $(d_{1,i}/d_{1,2})^6_{i=3}$.

$\mathbf{ip}_{13,C}[\mathbf{J} = [[]]] = \{\partial_x, \partial_y, y\partial_y, z\partial_z\}$: $z_x/z,\ z_{xx}/z,\ I_6,\ I_{56}$.

$\mathbf{ip}_{18,C}[s = 0, \mathbf{j} = [1]] = \{\partial_y, x\partial_y, z\partial_z, \theta_{0,1}(x)\partial_z\}$: $x,\ I_7,\ I_{59},\ I_{60}/z_y^3$.

Derived Series $(4, 0)$

$\mathbf{ip}_{18,A}[s = 2, t = 1] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y + \partial_z, \Psi_2(x)\partial_y + \varphi_2(x)\partial_z\}$:
$x,\ z_y,\ z_{yy},\ H_{63}$.

$\mathbf{ip}_{18,A}[s = t = 2] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, \Psi_2(x)\partial_y + \partial_z\}$: $x,\ z_y,\ z_{yy},\ H_{29}$.

$\mathbf{ip}_{18,A}[s = 2, t = 3] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, \Psi_2(x)\partial_y\}$: $x,\ z,\ z_y,\ z_{yy},\ H_{29}$.

$\mathbf{ip}_{18,C}[s = 1, \mathbf{j} = [0]] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y, z\partial_z\}$: $x,\ (I_i)^{65}_{i=63}$.

4.3.5 Groups with Three Generators

Derived Series (3)

$\mathbf{ip}_{7,A} = \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y + y^2\partial_z\}$:
$2z - yz_y$, $2z - y^2z_{yy}$, $z^2 - y^2z_x - yzz_y$, $2z^2 - yzz_y - y^3z_{xy} - y^2zz_{yy}$,
$2z^3 - 2y^2zz_x - 2yz^2z_y - y^4z_{xx} - 2y^3zz_{xy} - y^2z^2z_{yy}$.

$\mathbf{ip}_{8,A} = \{\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y + c(y-x)\partial_z\}$:
A^2z_{xy}, $(c+Az_x)(H_{51})_{a=-1}$, $(c+Az_y)(H_{51})_{a=1}$, $(c+Bz_{xx}+2Az_x)(H_{51})_{a=-2}$,
$(c - Bz_{yy} + 2Az_y)(H_{51})_{a=1}$, where $A := x - y$, $B := y^2 - 2yx + x^2$.

$\mathbf{ip}^1_{14,A} = \{\partial_y, y\partial_y + z\partial_z, y^2\partial_y + (2yz + xz^2)\partial_z\}$:
x, $z^2(1 + xz_y)/H^2_{55}$, $H_{56}/(zH_{55})$, $\{z_x^2 - z(2z_xz_y - zz_{xy})\}/H^2_{55}$,
$z^3\{x^3(zz_xz_{yy} - 2z_xz_y^2) + x^2(z^2z_{yy} - 6z_xz_y) - 2x(2z_x + zz_y) - 2z\}/H^4_{55}$.

$\mathbf{ip}^2_{14,A} = \{\partial_y, y\partial_y + z\partial_z, y^2\partial_y + (2yz + cz^2)\partial_z\}$:
x, $H_{56}/(zz_x)$, $(z/z_x)^2H_{57}$, zH_{58}/z_x^2, $(z/z_x)^3H_{59}$.

Derived Series $(3, 2, 0)$

$\mathbf{ip}^1_{1,A}[s = 0, t = 1] = \{\partial_y, \partial_x, x\partial_x + cy\partial_y + (c-1)z\partial_z\}$:
$z^{(2-c)/(c-1)}z_x$, $z^{1/(c-1)}z_y$, $z^{(3-c)/(c-1)}z_{xx}$, $z^{2/(c-1)}z_{xy}$, $z^{(c+1)/(c-1)}z_{yy}$.

$\mathbf{ip}^2_{1,A}[s = 0, t = 1] = \{\partial_y, \partial_x, x\partial_x + y\partial_y + c\partial_z\}$:
$z_x(H_{51})_{a=1}$, $z_y(H_{51})_{a=1}$, $z_{xx}(H_{51})_{a=2}$, $z_{xy}(H_{51})_{a=2}$, $z_{yy}(H_{51})_{a=2}$.

$\mathbf{ip}^2_{3,A}[s = 1] = \{\partial_y, \partial_x, x\partial_x + (y + x)\partial_y + c\partial_z\}$: H_{15}, H_{16}, H_{17}, H_{36}, H_{37}.

$\mathbf{ip}_{19,A}[s = 0] = \{\partial_y, y\partial_y + z\partial_z, x\partial_y\}$: x, z_y, zz_{yy}, H_9/z, H_{29}.

Derived Series $(3, 1, 0)$

$\mathbf{ip}_{13,A} = \{\partial_x, \partial_y, y\partial_y + \partial_z\}$: z_x, z_{xx}, e^zz_y, e^zz_{xy}, $e^{2z}z_{yy}$.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[1]]] = \{\partial_y, z\partial_z, \Psi_{1,1}(x)e^{c_1y}\partial_z\}$: x, $(e_{1,i}/e_{1,2})_{i=3,4,5,6}$.

$\mathbf{ip}_{10,C}[\mathbf{l} = []] = \{\partial_y, y\partial_y, z\partial_z\}$: x, z_x/z, z_{xx}/z, I_6, zI_{46}.

Derived Series $(3, 0)$

$\mathbf{ip}_{18,A}[s = 1, t = 1] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y + \partial_z\}$: x, z_y, z_{yy}, H_{29}, H_{64}.

$\mathbf{ip}_{18,A}[s = 1, t = 2] = \{\partial_y, x\partial_y, \Psi_1(x)\partial_y\}$: x, z, z_y, z_{yy}, H_{29}.

$\mathbf{ip}_{18,C}[s = 0, \mathbf{j} = [0]] = \{\partial_y, x\partial_y, z\partial_z\}$: x, $(I_i)_{i=63}^{66}$.

4.3.6 Groups with Two Generators

Derived Series $(2, 1, 0)$

$\mathbf{ip}_{10,A} = \{\partial_x, y\partial_y + \partial_z\}$: x, z_x, z_{xx}, $e^z z_y$, $e^z z_{xy}$, $e^{2z} z_{yy}$.

$\mathbf{ip}_{12,A} = \{\partial_x, x\partial_x + y\partial_y + c\partial_z\}$: $z - c\log(y)$, yz_x, yz_y, $y^2 z_{xx}$, $y^2 z_{xy}$, $y^2 z_{yy}$.

Derived Series $(2, 0)$

$\mathbf{ip}_{11,A} = \{\partial_x, \partial_y\}$: z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}.

$\mathbf{ip}_{18,A}[s = 0] = \{\partial_y, x\partial_y\}$: x, z, z_y, z_{yy}, H_9, H_{29}.

$\mathbf{ip}_{9,C}[\mathbf{L} = [[0]]] = \{\partial_y, z\partial_z\}$: x, z_x/z, z_y/z, z_{xx}/z, z_{xy}/z, z_{yy}/z.

4.3.7 Groups with One Generator

Derived Series $(1, 0)$

$\mathbf{ip}_{9,A} = \{\partial_y\}$: x, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}.

4.3.8 Groups whose Invariants were not found

Derived Series (6)

$\mathbf{p}_5 = \{\partial_x, \partial_y, \partial_z, (u\partial_v - v\partial_u)_{(u,v)=(x,y),(x,z),(y,z)}\}$.

Derived Series $(n, n-1, 0)$

$\mathbf{ip}_{20,A}[3 \leq 1 + q + \sum_{i=1}^{q} s_i \leq 5]$: z_y, z_{yy}, remaining invariants huge.

$\mathbf{ip}_{21,A}[3 \leq 3 + h + s + \sum_{i=1}^{h} s_i \leq 6]$: z_y, remaining invariants huge.

Chapter 5

Lower Invariants

In this chapter, we deal with groups with more than seven parameters. Included are all types of space groups, except Amaldis groups of type B. We usually indicate the lower invariants; in those exceptional cases, where the coefficient matrix of the determining system has rank less than eight, we indicate the invariant basis.

The invariants in this chapter are listed lexicographically according to the derived series of their corresponding groups. To this end, the structure of any derived series is represented as a function of the unknown group size, which is always denoted by n.

Example: $\mathbf{ip}_{18}[h = 1]$ has the derived series $(9, 8)$. Its lower invariants are found in Section 5.2 ("Lie's Imprimitive Space Groups") in the list "Derived Series $(n, n - 1)$".

The groups themselves are indicated by the names defined in Chapter 3 ("The Space Point Groups"). For example, the $\mathbf{p}_i$ appearing in the lower invariant list for Lie's primitive space groups are defined in Section 3.3 ("The Primitive Space Groups").

As usual, $[\cdot]$ denotes the truth value function $[x] = 1$, if x is true, $[x] = 0$ otherwise. If x is not a truth value formula, the symbols '[', ']' simply denote square brackets.

5.1 Lie's Primitive Space Groups

The expressions F_i can be found in Subsection 4.2.1.

5.1.1 Lower Invariants

Derived Series (n)

$\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$: F_2.

$\mathbf{p}_4$: F_2, $F_3 + F$,

$$F := z_{xx}(x^2 + 2xz_y) + 2z_{xy}(xy - xz_x + yz_y) + z_{yy}(y^2 - 2yz_x).$$

$\mathbf{p}_8$: F_1, $F_3^2 + 2(F_2 + z_{xy}^2)F_1 + F$,

$$F := z_{xx}^2(1 + 2z_y^2) + z_{yy}^2(1 + 2z_x^2) - 4z_x z_y z_{xy}(z_{xx} + z_{yy}).$$

5.2 Lie's Imprimitive Space Groups

The expressions G_i can be found in Subsection 4.2.1.

5.2.1 Lower Invariants

Derived Series (n)

$\mathbf{ip}_{30}[h = 1]$, $\mathbf{ip}_{32}[h = 1]$: G_2.

$\mathbf{ip}_{11}[m = 1]$, $\mathbf{ip}_{12}[m = 1]$: G_2, G_3.

$\mathbf{ip}_{10}(= \mathbf{ip}_{33})$, $\mathbf{ip}_{11}[m \neq 1]$, $\mathbf{ip}_{12}[m > 1]$: G_3.

$\mathbf{ip}_{17}[h > 1]$, $\mathbf{ip}_{30}[h > 1]$, $\mathbf{ip}_{32}[h > 1]$: none.

$\mathbf{ip}_{31}$: z, G_2.

Derived Series $(n, n-1)$

$\mathbf{ip}_4$, $\mathbf{ip}_5[\mathbf{m} \notin \{[0], [1], [0, 0]\}]$, $\mathbf{ip}_{12}[m = 0](= \mathbf{ip}_{26})$, $\mathbf{ip}_{30}[h = 0]$: G_3.

$\mathbf{ip}_{18}[h = 1]$, $\mathbf{ip}_{23}[h = 1]$: G_2.

$\mathbf{ip}_{18}[h > 1]$, $\mathbf{ip}_{23}[h > 1]$: none.

Derived Series $(n, n-1, n-2)$

$\mathbf{ip}_7$, $\mathbf{ip}_8[m > 1]$, $\mathbf{ip}_{32}[h = 0]$: G_3.

$\mathbf{ip}_8[m = 1]$: G_2, G_3.

Derived Series $(n, n-2)$

$\mathbf{ip}_6[\mathbf{m} = [1]]$: G_2, G_3.

$\mathbf{ip}_6[\mathbf{m} = [0, 0]]$: G_3, $G_2 + \lambda_2 G_3$.

$\mathbf{ip}_6[\mathbf{m} \notin \{[\mathbf{0}], [\mathbf{1}], [\mathbf{0}, \mathbf{0}]\}]$: G_3.

$\mathbf{ip}_{25}[m = 1]$: G_2.

$\mathbf{ip}_{25}[m > 1]$: none.

Derived Series $(n, n-2, n-3)$

$\mathbf{ip}_9[m \in \{0, 1\}](= \mathbf{ip}_{25}[h = 0])$: G_2, G_3.

$\mathbf{ip}_9[m > 1]$: G_3.

5.2.2 Rank Seven

Derived Series (n)

$\mathbf{ip}_1(= \mathbf{ip}_{28})$, $\mathbf{ip}_2[l > 2]$: z.

$\mathbf{ip}_{17}[h = 1]$: G_2.

$\mathbf{ip}_{27}$: A^2/B^3, where A and B are defined at the end of the chapter.

$\mathbf{ip}_{29}$: $e^{4z}(G_2 - 2G_3/3)$.

Derived Series $(n, n-1)$

$\mathbf{ip}_3[l > 1]$: z.

$\mathbf{ip}_5[\mathbf{m} = [1]]$: $G_2/G_3^{4/3}$.

$\mathbf{ip}_5[\mathbf{m} = [0, 0]]$: $(G_2 + \lambda_2 G_3)/G_3^{4/3}$.

5.3 Amaldis Groups Of Type A

The expressions H_i can be found in Subsection 4.2.2.

5.3.1 Lower Invariants

Derived Series (n)

$\mathbf{ip}_{5,A}[s > 2]$: z_{yy}, H_{46}.

Derived Series $(n, n-2, n-4, 0)$

$\mathbf{ip}_{4,A}[s > 3, 1 < t < s]$: z_y, z_{yy}.

5.3.2 Rank Seven

Derived Series (n)

$\mathbf{ip}^1_{2,A}[s > 3]$: H_{21}.

Derived Series $(n, n-1, n-3, 0)$

$\mathbf{ip}^1_{1,A}[s > 4, 1 < t < s]$: H_1.

$\mathbf{ip}^2_{1,A}[s > 4, 1 < t < s]$: H_{11}.

$\mathbf{ip}^1_{3,A}[s > 5, 1 < t < s-1]$: H_{26}.

Derived Series $(n, n-1, 0)$

$\mathbf{ip}_{21,A}[3 + h + s + \sum_{i=1}^{h} s_i > 7]$: z_y.

Derived Series $(n, n-2, n-4, 0)$

$\mathbf{ip}_{4,A}[s > 3, t = 1]$: H_2.

$\mathbf{ip}_{4,A}[s > 3, t = s]$: H_{19}.

Derived Series $(n, n-2, 0)$

$\mathbf{ip}_{21,A}[3 + s + h + \sum_{i=1}^{h} s_i > 7]$: z_y.

5.3.3 Rank Six

Derived Series (n)

$\mathbf{ip}^2_{2,A}[s > 3]$: H_{23}, H_{24}.

Derived Series $(n, n-1, n-3, 0)$

$\mathbf{ip}^1_{1,A}[s > 4, t = 1]$: H_1, H_2.

$\mathbf{ip}^1_{1,A}[s > 4, t = s]$: H_1, H_8.

$\mathbf{ip}^2_{1,A}[s > 4, t = 1]$: H_{11}, H_{12}.

$\mathbf{ip}^1_{3,A}[s > 5, t = 1]$: H_2, H_{26}.

$\mathbf{ip}^1_{3,A}[s > 5, t = s-1]$: H_{26}, H_{30}.

Derived Series $(n, n-1, 0)$

$\mathbf{ip}_{19,A}[s > 4, t = 1, .., s-1]$: x, z_y.

$\mathbf{ip}_{20,A}[1 + q + \sum_{i=1}^{q} s_i > 7]$: z_y, z_{yy}.

Derived Series $(n, n-2, n-4, 0)$

$\mathbf{ip}^2_{1,A}[s > 4, t = s]$: H_{11}, H_{19}.

$\mathbf{ip}_{4,A}[s > 3, t = s+1]$: H_{19}, H_{41}.

5.3.4 Rank Five

Derived Series $(n, n-1, n-3, 0)$

$\mathbf{ip}^1_{1,A}[s > 4, t > s]$: H_5, H_6, H_7.

$\mathbf{ip}^2_{1,A}[s > 4, t > s]$: H_{15}, H_{16}, H_{17}.

$\mathbf{ip}^2_{3,A}[s > 5]$: H_{15}, H_{16}, H_{17}.

Derived Series $(n, n-1, 0)$

$\mathbf{ip}_{19,A}[s > 4, t = s]$: x, z_y, H_{29}.

Derived Series $(n, 0)$

$\mathbf{ip}_{18,A}[s > 5, t = 1, \ldots, s-1]$: x, z_y, z_{yy}.

5.3.5 Rank Four

Derived Series $(n, 0)$

$\mathbf{ip}_{18,A}[s > 5, t = s]$: x, z_y, z_{yy}, H_{29}.

Derived Series $(n, n-1, 0)$

$\mathbf{ip}_{19,A}[s > 4, t = s+1]$: x, z_y, zz_{yy}, H_{29}.

5.3.6 Rank Three

Derived Series $(n, 0)$

$\mathbf{ip}_{18,A}[s > 5, t = s+1]$: x, z, z_y, z_{yy}, H_{29}.

5.4 Amaldis Groups Of Type C

The expressions I_i can be found in Subsection 4.2.3.

5.4.1 Group Parameter Related Numbers

In this section, some of the derived series in the headings contain parameters like d, e, etc.

Example: "Derived Series $(n, n-2, k, 0)$" indicates that the groups listed below that heading have a derived series of the form $(n, n-2, k, 0)$ for some k with $0 \leq k \leq n-2$.

These parameters are actually linked to the parameters of a corresponding group, a complete list of these correspondences is given below.

For $\mathbf{ip}_{1,C}(s, \mathbf{m} = [m_0, \dots, m_l], m_{j+1} \geq m_j + s)$ we define

$$d := \sum_{i=0}^{l-1} [m_{i+1} - m_i > s], \text{ and}$$

$$e := [s \neq 0] \cdot \max(0, \sum_{i=0}^{l-1} m_i + (s-1)(l - [m_0 = 0]) + \sum_{i=0}^{l-2} [m_{i+1} - m_i = s]).$$

For $\mathbf{ip}_{3,C}(s, \mathbf{m} = [m_0, \dots, m_l], m_{j+1} \geq m_j + s)$ we define

$$f := \max(0, \sum_{i=0}^{l-1} m_i + (s-2)(l - [m_0 = 0])).$$

For $\mathbf{ip}_{10,C}[\mathbf{l} = [l_0, \dots, l_m]]$ we define

$$g := \sum_{i=1}^{m} l_i.$$

For $\mathbf{ip}_{12,C}[i = [i_1, , i_m], \mathbf{U} = [[(u_{l,i})_{i=1}^{i_m}]_{l=0}^{m}]]]$ we define

$$k := \sum_{l=1}^{m} (i_l + \sum_{i} u_{l,i}).$$

For $\mathbf{ip}_{13,C}[\mathbf{J} = [[(j_{m,i})_{i=0}^{i_m}]_{m=1}^{s}]]$ we define

$$j := s + \sum i_m.$$

For $\mathbf{ip}_{19,C}[s, \mathbf{j} = [j_0, \dots, j_m]]$ we define

$$h := \sum_{k=0}^{m} j_k s_k,$$

where $(s_k)_{k \geq 0}$ is the unique solution of the initial value difference sequence problem

$$\Delta_0^{(0)} = s_0 := 0, \; (\Delta_0^{(i)} := \binom{s+2}{i})_{i=1}^{s+1}, \; \Delta^{(s+2)} \equiv 1.$$

Remark: Note that a derived series of the form $(n, n-2, n-5-d, e, 0)$ may contain cases like $(n, n-2, n-5-d, 0, 0) = (n, n-2, n-5-d, 0)$, and $(n, n-2, 0, 0, 0) = (n, n-2, 0)$, too. This implies that there might be alternative possibilities to look after a group with derived series $(n, n-2, 2)$.

5.4.2 Lower Invariants

Derived Series (n)

$\mathbf{ip}_{2,C}[s = 1, m_0 = 0, l = 1]$: z_{yy}, I_{11}.

$\mathbf{ip}_{2,C}[s > 0, m_0 > \max(0, 2 - s), l = 0]$: z_y, z_{yy}.

$\mathbf{ip}_{2,C}[s > 0, m_0 \geq \max(0, 2 - s), l = 1]$: z_{yy}.

$\mathbf{ip}_{2,C}[s > 0, m_0 \geq 0, l > 1]$: none.

$\mathbf{ip}_{5,C}[s = 1, m_0 = 0, l = 1]$: z_{yy}, I_{11}.

$\mathbf{ip}_{5,C}[s = 1, m_0 = 1, l = 0]$: z_y, z_{yy}, I_{11}.

$\mathbf{ip}_{5,C}[s > 0, m_0 > \max(0, 2 - s), l = 0]$: z_y, z_{yy}.

$\mathbf{ip}_{5,C}[s > 0, m_0 \geq \max(0, 2 - s), l = 1]$: z_{yy}.

$\mathbf{ip}_{5,C}[s > 0, m_0 \geq 0, l > 1]$: none.

$\mathbf{ip}_{6,C}[\mathbf{m} = [0, 1]]$: y, I_{32}, I_{33}.

$\mathbf{ip}_{6,C}[l = 1, \mathbf{m} \neq [0, 1]]$: y, I_{32}.

$\mathbf{ip}_{6,C}[l > 1]$: y.

$\mathbf{ip}_{17,C}[m = 0, n > 0]$: z_x, $[n = 1]z_{yy}$.

$\mathbf{ip}_{17,C}[m > 0, n = 0]$: z_y, $[m = 1]z_{xx}$.

$\mathbf{ip}_{17,C}[m, n > 0]$: $[m = 1]z_{xx}$, $[n = 1]z_{yy}$.

Derived Series $(n, n - 1)$

$\mathbf{ip}_{2,C}[s = 0, m_0 = l = 1]$: z_{xx}, z_{yy}.

$\mathbf{ip}_{2,C}[s = 0, m_0 > 1, l = 1]$: z_{yy}.

$\mathbf{ip}_{2,C}[s = 0, m_0 = 1, l > 1]$: z_{xx}.

$\mathbf{ip}_{2,C}[s = 0, \min(m_0, l) > 1]$: none.

$\mathbf{ip}_{15,C}[p > 0, \mathbf{m} = [1]]$: e^{ax}, $[p = 1]z_{yy}$, I_{54}.

$\mathbf{ip}_{15,C}[p > 0, \mathbf{m} = [m], m > 1]$: e^{ax}, $[p = 1]z_{yy}$.

$\mathbf{ip}_{15,C}[p > 0, \mathbf{m} = [0, 0]]$: e^{ax}, $[p = 1]z_{yy}$, I_{55}.

$\mathbf{ip}_{15,C}[p > 0, k > 1, \mathbf{m} \neq [0, 0]]$: e^{ax}, $[p = 1]z_{yy}$.

Derived Series $(n, n-1, n-2)$

$\mathbf{ip}_{2,C}[s > 1, m_0 = l = 0]$: z_y, z_{yy}.

$\mathbf{ip}_{5,C}[s = 0, m_0 = 1, l = 0]$: z_y, $z_{xx,}$ z_{yy}.

$\mathbf{ip}_{5,C}[s = 0, m_0 = 1, l = 1]$: $z_{xx,}$ z_{yy}.

$\mathbf{ip}_{5,C}[s = 0, m_0 = 1, l > 1]$: z_{xx}..

$\mathbf{ip}_{5,C}[(s = 0, m_0 > 1, l = 0) \vee (s > 1, m_0 = l = 0)]$: z_y, z_{yy}.

$\mathbf{ip}_{5,C}[s = 0, m_0 > 1, l = 1]$: z_{yy}.

$\mathbf{ip}_{5,C}[s = 0, m_0 > 1, l > 1]$: none.

$\mathbf{ip}_{5,C}[s = 1, m_0 = l = 0]$: z_y, z_{yy}, I_{10}.

$\mathbf{ip}_{7,C}[m > 1]$: y.

$\mathbf{ip}_{16,C}[l = 1, m = 0]$: $[m = 0]z_x$, $[m \leq 1]z_{xx}$, z_{yy}.

$\mathbf{ip}_{16,C}[l > 1, m = 0]$: $[m = 0]z_x$, $[m \leq 1]z_{xx}$.

$\mathbf{ip}_{17,C}[m = n = 0]$: z_x, z_y, z_{xy}.

Derived Series $(n, n-2, n-4, 3)$

$\mathbf{ip}_{5,C}[s = m_0 = 0, l = 1]$: z_x, $z_{xy,}$ z_{yy}.

$\mathbf{ip}_{5,C}[s = m_0 = 0, l > 1]$: z_x, z_{xy}.

$\mathbf{ip}_{16,C}[l = 0, m = 1]$: z_y, z_{xx}, z_{xy}.

$\mathbf{ip}_{16,C}[l = 0, m > 1]$: z_y, z_{xy}.

Derived Series $(n, n-2, k, 0)$

$\mathbf{ip}_{12,C}[m > 0, \mathbf{U}_0 \neq [0], (m = 1 \vee \mathbf{U}_0 = [1]) \Rightarrow \mathbf{U}_1 \neq [0], \sum i_l + \sum u_{l,i} > 4]$:
x^c, $[m = 1]z_{yy}$, $[\mathbf{U}_1 = [0]]I_{86}$, $[\mathbf{U}_0 = \mathbf{U}_1 = [1]]I_{87}$.

Derived Series $(n, n-2, n-4-s-l, 0)$

$\mathbf{ip}_{3,C}[s \in \{0,1\}, \mathbf{m} = [m_0, m_1], m_0 > -s, m_0 + m_1 > 2 - s]$: z_{yy}.

$\mathbf{ip}_{3,C}[s = 0, l > 1, m_{l-1} = 1, m_l = 1]$: z_{xx}.

$\mathbf{ip}_{3,C}[s = 0, l > 1, m_{l-1} = 0, m_l > 1]$: z_{xy}.

$\mathbf{ip}_{3,C}[s \in \{0,1\}, l > 1, m_{l-1} > 0, m_l > 1]$: none.

Derived Series $(n, n-2, n-5-l, f, 0)$

$\mathbf{ip}_{3,C}[s > 1, \mathbf{m} = [m], m > \max(0, 3-s)]$: z_y, z_{yy}.

$\mathbf{ip}_{3,C}[s > 1, l = 1]$: z_{yy}.

$\mathbf{ip}_{3,C}[s > 1, l > 1]$: none.

Derived Series $(n, n-2, n-5-d, e, 0)$

$\mathbf{ip}_{1,C}[s = 0, \mathbf{m} = [0, m], m > 1]$: z_{xy}, z_{yy}.

$\mathbf{ip}_{1,C}[s = 0, \mathbf{m} = [1, 1]]$: z_{xx}, z_{yy}.

$\mathbf{ip}_{1,C}[(s = 0, l = 1, m_0 > 0, m_1 > 1) \vee (s > 0, l = 1, \mathbf{m} \neq [0, 1])]$: z_{yy}.

$\mathbf{ip}_{1,C}[s = 0, l > 1, m_{l-1} = 0, m_l = 1]$: z_{xx}, z_{xy}.

$\mathbf{ip}_{1,C}[s = 0, l > 1, m_{l-1} = 0, m_l > 1]$: z_{xy}.

$\mathbf{ip}_{1,C}[s = 0, l > 1, m_{l-1} = m_l = 1]$: z_{xx}.

$\mathbf{ip}_{1,C}[(s = 0, l > 1, m_{l-1} > 0, m_l > 1) \vee (s > 0, l > 1)]$: none.

$\mathbf{ip}_{1,C}[s > 0, \mathbf{m} = [m], m \geq \max(3-s, 1)]]$: z_y, z_{yy}.

$\mathbf{ip}_{1,C}[s = 1, \mathbf{m} = [0, 1]]$: z_{yy}, I_{11}.

Derived Series $(n, n-3, n-4-j, 0)$

$\mathbf{ip}_{13,C}[s=1, i_1>0, i_1+\sum j_{1,i}>2, \{j_{1,i}\} \neq \{0\}]$:
e^{cx}, $[j_{1,0}=1]z_{yy}$, $[i_1=1]I_{54}$, $[j_{1,1}=0]I_{73}$.

$\mathbf{ip}_{13,C}[((s=2, \sum i_m + \sum j_{m,0} > 1) \vee (s>2)), \{j_{m,i}\} \neq \{0\}]$:
e^{cx}, $[\{j_{m,i}\} \subseteq \{0,1\}]z_{yy}$, $[s=2, i_1=i_2=0]I_{55}$,
$[\exists! M : j_{M,0}=a>0, \{j_{m,i}\}_{(M,0)}^{(m,i)\neq} = \{0\}]I_{77}$.

Derived Series $(n, n-3, n-6-d, e, 0)$

$\mathbf{ip}_{4,C}[s=0, \mathbf{m}=[m], m>1]$: z_y, z_{xy}, z_{yy}.

$\mathbf{ip}_{4,C}[s=0, \mathbf{m}=[0,m], m>1]$: z_{xy}, z_{yy}.

$\mathbf{ip}_{4,C}[s=0, \mathbf{m}=[1,1]]$: z_{xx}, z_{yy}.

$\mathbf{ip}_{4,C}[s \geq 0, l=1, s=0 \Rightarrow m_0>0, \mathbf{m} \neq [1,1], s=1 \Rightarrow \mathbf{m} \neq [0,1]]$: z_{yy}.

$\mathbf{ip}_{4,C}[s=1, \mathbf{m}=[1]]$: z_y, z_{yy}, I_{11}.

$\mathbf{ip}_{4,C}[s>0, \mathbf{m}=[m], m>\max(0, 2-s)]$: z_y, z_{yy}.

$\mathbf{ip}_{4,C}[s=1, \mathbf{m}=[0,1]]$: z_{yy}, I_{11}.

$\mathbf{ip}_{4,C}[s>1, \mathbf{m}=[0]]$: z_y, z_{yy}, I_9.

$\mathbf{ip}_{4,C}[s=0, l>1, m_l=0]$: z_x, z_{xx}, z_{xy}.

$\mathbf{ip}_{4,C}[s=0, l>1, m_{l-1}=0, m_l=1]$: z_{xx}, z_{xy}.

$\mathbf{ip}_{4,C}[s=0, l>1, m_{l-1}=0, m_l>1]$: z_{xy}.

$\mathbf{ip}_{4,C}[s=0, l>1, m_{l-1}=1, m_l=1]$: z_{xx}.

$\mathbf{ip}_{4,C}[(s=0, l>1, m_{l-1}>0, m_l>1) \vee (s>0, l>1)]$: none.

Derived Series $(n, n-3, 0)$

$\mathbf{ip}_{11,C}[p=1, k_1=1, l_{1,0}>1, l_{1,1}>0]$: e^{ax+by}, $2a_1d_{1,2}-d_{1,4}$.

$\mathbf{ip}_{11,C}[p=1, k_1>1, l_{1,0}+l_{1,1}>1]$:
e^{ax+by}, $[l_{1,0}=1](2b_1d_{1,3}-d_{1,6})$, $[l_{1,1}=0](a_1^2d_{1,3}-d_{2,5})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[l_{1,0},0],[0]], l_{1,0}>1]$: e^{ax+by}, $a_1^2a_2d_{1,2,3}-d_{2,4,5}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,1],[0]]]$: e^{ax+by}, $d_{1,4,6}-2a_1d_{1,2,6}+2b_1d_{1,3,4}+4a_1b_1d_{1,2,3}$.

$\mathbf{ip}_{11,C}[\mathbf{k}=[0,0], l_{1,0}, l_{2,0}>0, l_{1,0}+l_{2,0}>2]$: e^{ax+by}, $a_1a_2d_{1,2}-d_{2,4}$.

$\mathbf{ip}_{11,C}[\mathbf{k}=[k_1,0], k_1>1, \sum l_{1,k}\geq 1, l_{2,0}=0]$:
e^{ax+by}, $[\sum l_{1,k}=1](b_1^2b_2d_{1,2,3}-d_{3,5,6})$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,0],\mathbf{l}_2], \mathbf{l}_2\in\{[1],[0,0],[1,0]\}]$:
e^{ax+by}, $2(b_1+b_2)(a_1b_2-a_2b_1)d_{1,2,3}-(b_2-b_1)d_{1,4,6}+2(a_2-a_1)d_{1,5,6}$.

$\mathbf{ip}_{11,C}[\mathbf{k}=[k_1,0], k_1, l_{2,0}>0, k_1+l_{2,0}>2, \sum l_{1,k}=0]$: e^{ax+by}, $a_1a_2d_{1,3}-d_{2,5}$.

$\mathbf{ip}_{11,C}[\mathbf{k}=[k_1,k_2], k_1k_2(k_1+k_2)>2, \sum l_{i,k}=0]$: e^{ax+by}, $b_1b_2d_{1,3}-d_{3,6}$.

$\mathbf{ip}_{11,C}[\mathbf{k}=[k_1,0,0], k_1>1, \sum l_{i,k}=0]$:
e^{ax+by}, $[a_2(b_1+b_3)-a_3(b_1+b_2)]d_{1,2,3}-(b_3-b_2)d_{1,2,5}+(a_3-a_2)d_{1,2,6}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[l_{1,0}],[0],[0]], l_{1,0}>1]$:
e^{ax+by}, $a_1(a_3-a_2)d_{1,2,3}+(b_3-b_2)d_{1,2,4}+(a_3-a_2)d_{1,2,5}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1],[1],[0]]]$: $(a_3-a_2)(a_3-a_1)[(a_2-a_1)d_{1,2,3,6}-2(b_2-b_1)d_{1,2,3,5}]$
$+\sum_{l\to 3}^{i,j\in\{1,2,3\}\backslash l, i>j}(-1)^l(a_i-a_j)(b_l^2-2b_3b_l)d_{1,2,3,4}$, e^{ax+by}.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1,0],[0],[0]]]$:
$(b_2-b_1)(b_3-b_1)d_{1,2,3,4}+(a_2-a_1)(a_3-a_1)d_{1,2,3,6}$
$-[(a_2-a_1)(b_3-b_1)+(a_3-a_1)(b_2-b_1)]d_{1,2,3,5}$, e^{ax+by}.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[1],[0]]]$:
e^{ax+by}, $(a_3-a_2)(b_3-b_1)\{(b_2-b_1)^2d_{1,2,3,4}+(a_2-a_1)^2d_{1,2,3,6}\}-$
$[a_1^2(b_3-b_2)^2+a_2(a_2-2a_1)(b_3-b_1)(b_1+b_3-2b_2)+a_3(a_3-2a_1)(b_2-b_1)^2]d_{1,2,3,5}$.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[0,0],[0]]]$: $(b_3-b_1)(b_3-b_2)[(b_2-b_1)d_{1,2,3,4}-2(a_2-a_1)d_{1,2,3,5}]$ $+[(a_3-a_2)^2b_1-(a_3-a_1)^2b_2-(a_2-a_1)(a_1+a_2-2a_3)b_3]d_{1,2,3,6}$, e^{ax+by}.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[1],[0],[0],[0]]]$: $(\sum_{l=1}^{4}(-1)^l(b_l^2-2b_1b_l)\prod_{i,j\in\bar{4}\setminus\{l\}}^{i>j}(a_i-a_j))d_{1,2,3,4,5}-$ $-\prod_{l=2}^{4}(a_l-a_1)\sum_{l\in\{2,3,4\}}^{i,j\in\{2,3,4\}\setminus l,i>j}(-1)^l(a_i-a_j)b_ld_{1,2,3,4,6}$, e^{ax+by}.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0,0],[0],[0],[0]]]$:
$(\sum_{l=1}^{4}(-1)^l(a_l^2-2a_1a_l)\prod_{i,j\in\bar{4}\setminus\{l\}}^{i>j}(b_i-b_j))d_{1,2,3,4,5}-$
$-\prod_{l=2}^{4}(b_l-b_1)\sum_{l\in\{2,3,4\}}^{i,j\in\{2,3,4\}\setminus l,i>j}(-1)^l(b_i-b_j)a_ld_{1,2,3,4,6}$, e^{ax+by}.

$\mathbf{ip}_{11,C}[\mathbf{L}=[[0],[0],[0],[0],[0]]]$: e^{ax+by}, $d_{1,2,3,4,5,6}$.

$\mathbf{ip}_{11,C}$[rank 8; $\mathbf{L}$ other than above]: e^{ax+by}.

5.4.3 Rank Seven

Derived Series (n)

$\mathbf{ip}_{2,C}[s=m_0=1,l=0]$: $I_{11}/(z_{yy}I_7)^2$.

$\mathbf{ip}_{6,C}[\mathbf{m}=[m],m>1]$: I_{29}.

$\mathbf{ip}_{8,C}[m>2]$: I_{28}.

$\mathbf{ip}_{14,C}[m>0,l>1]$: x.

Derived Series $(n,n-1)$

$\mathbf{ip}_{2,C}[s=0,m_0>1,l=0]$: I_7.

$\mathbf{ip}_{15,C}[p>1,\mathbf{m}=[0]]$: I_{53}.

Derived Series $(n,n-2,n-5,0)$

$\mathbf{ip}_{1,C}[s=0,\mathbf{m}=[m],m>2]$: I_7/I_6^c.

$\mathbf{ip}_{1,C}[s=0,\mathbf{m}=\mathbf{0}_l,n>3]$: I_3/I_2^c.

$\mathbf{ip}_{1,C}[s>2,\mathbf{m}=[0]]$: $I_9/(z_y^2I_7^{1/c})$.

$\mathbf{ip}_{3,C}[s=0,\mathbf{m}=[0,m],m>2]$: $(I_{15})_{i=14}$.

$\mathbf{ip}_{3,C}[s=1,\mathbf{m}=[m],m>2]$: I_{18}.

$\mathbf{ip}_{3,C}[s>3,\mathbf{m}=[0]]$: I_{20}.

$\mathbf{ip}_{12,C}[m>2,\mathbf{U}_0=[1],\mathbf{U}_1=[0]]$: xI_{86}/I_{84}.

$\mathbf{ip}_{12,C}[m>1,i_0=2,i_1=1,\{u_{l,i}\}=\{0\}]$: xI_{86}/I_{85}.

Derived Series $(n,n-2,g,0)$

$\mathbf{ip}_{10,C}[m>0,l_0>1,\sum_{i=0}^{m} l_i>4]$: x.

Derived Series $(n,n-2,h,0)$

$\mathbf{ip}_{19,C}[s\geq 0,l>0,s=0\Rightarrow \mathbf{j}\neq[0,1]]$: x.

Derived Series $(n,n-2,3)$

$\mathbf{ip}_{2,C}[s=m_0=0,l>1]$: I_3.

$\mathbf{ip}_{15,C}[p=0,\|\mathbf{m}\|_1+k>2]$: I_6.

Derived Series $(n,n-2,1,0)$

$\mathbf{ip}_{3,C}[s=0,l>2,m_{l-1}=0,m_l=1]$: $(I_{15})_{i=12}$.

$\mathbf{ip}_{12,C}[m=1,\mathbf{U}_1=[0],i_0+\sum u_{0,i}>3]$: xz_{yy}/I_{86}.

Derived Series $(n,n-2,0)$

$\mathbf{ip}_{9,C}[\text{size}>7, \text{rank } 7]$: x.

$\mathbf{ip}_{19,C}[s\geq 0,\mathbf{j}=[j],j>\max(1,3-s)]$: x.

Derived Series $(n,n-3,n-5,0)$

$\mathbf{ip}_{13,C}[s=1,i_1=0,j_{1,0}>2]$: I_{53}.

Derived Series $(n, n-3, n-7, 0)$

$\mathbf{ip}_{4,C}[s = 0, \mathbf{m} = [0, 1]]$: I_{22}.

Derived Series $(n, n-3-s, 0)$

$\mathbf{ip}_{18,C}[s \geq 0, l > 0, s = 0 \Rightarrow \mathbf{j} \notin \{[0, 1], [1, 1]\}]$: x.

Derived Series $(n, n-3, 0)$

$\mathbf{ip}_{11,C}[p = 1, k_1 = 1, l_{1,0} > 2, l_{1,1} = 0]$: $(a_1^2 d_{1,3} - d_{2,5})/(a_1^2 d_{1,2} - d_{2,4})$.

$\mathbf{ip}_{11,C}[p = 1, k_1 > 2, l_{1,0} = 1, l_{1,1} = 0]$: $(b_1^2 d_{1,3} - d_{3,6})/(a_1^2 d_{1,3} - d_{2,5})$.

$\mathbf{ip}_{11,C}[p = 2, k_1 = k_2 = 0, l_{1,0} > 2, l_{2,0} = 0]$: $(b_1 b_2 d_{1,2} - d_{3,5})/(a_1 a_2 d_{1,2} - d_{2,4})$.

$\mathbf{ip}_{11,C}[p = 2, k_1 > 2, k_2 = 0, \sum l_{1,k} = 0, l_{2,0} = 0]$:
$(b_1 b_2 d_{1,3} - d_{3,6})/(a_1 a_2 d_{1,3} - d_{2,5})$.

$\mathbf{ip}_{13,C}[s \geq 1, s + \sum i_m > 3, \{j_{m,i}\} = \{0\}]$: I_6.

5.4.4 Rank Six

Derived Series (n)

$\mathbf{ip}_{14,C}[m > 2, l = 1]$: x, I_{48}.

Derived Series $(n, n-1, 3)$

$\mathbf{ip}_{14,C}[m = 0, l > 3]$: x, I_6.

Derived Series $(n, n-2, n-4)$

$\mathbf{ip}_{3,C}[s = 0, \mathbf{m} = [m], m > 3]$: $(I_{16})_{i=6,7}^{a=7}$.

Derived Series $(n, n-2, n-4, 0)$

$\mathbf{ip}_{12,C}[m > 3, \mathbf{U}_0 = [0]]$: I_{81}/I_{78}, xI_{86}/I_{78}.

Derived Series $(n, n-2, m-1, 0)$

$\mathbf{ip}_{10,C}[m > 4, l_0 = 1]$: x, I_{47}/I_{48}.

Derived Series $(n, n-2, 0)$

$\mathbf{ip}_{3,C}[s=0, l>3, m_l=0]$: $(I_{16})_{i=2,3}^{a=3}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[l,1]], l>4]$: x, $(c_1^2 e_{1,3}-e_{3,6})/(c_1^2 e_{1,2}-e_{3,5})$.

$\mathbf{ip}_{9,C}[s=1, m_1>3, l_{1,0}=2, l_{1,1}=1]$: x, $e_{1,2,3,5}/e_{1,2,4}$.

$\mathbf{ip}_{9,C}[\mathbf{L}=[[l],[1]], l>4]$: x, $(c_2 e_{1,3}-e_{3,6})/(\Psi'_{2,1}e_{1,3}-\Psi_{2,1}e_{2,5})$.

$\mathbf{ip}_{9,C}[\mathbf{m}=[m,0], m>3, l_{i,j}=1]$:
x, $(c_2 e_{1,2}-e_{3,5})/(\Psi''_{2,1}e_{1,2}-\Psi'_{2,1}e_{1,4}+\Psi_{2,1}e_{2,4})$.

$\mathbf{ip}_{10,C}[m=0, l_0>4]$: x, I_6.

$\mathbf{ip}_{12,C}[m=0, i_0+\sum u_{0,i}>4]$: $(I_i)_{i=82}^{83}$.

$\mathbf{ip}_{19,C}[s>2, \mathbf{j}=[1]]$: x, I_{59}.

Derived Series $(n, n-3, 0)$

$\mathbf{ip}_{11,C}[p=1, k_1=0, l_{1,0}>3]$: $I/d_{1,2}$, $I\in\{d_{1,4}, d_{3,5}\}$.

$\mathbf{ip}_{11,C}[p=1, k_1>3, l_{1,0}=0]$: $(d_{2-i,5+i}/d_{1,3})_{i=0}^{1}$.

Derived Series $(n, n-3-s, 0)$

$\mathbf{ip}_{18,C}[s\geq 0, \mathbf{j}=[j], j>\max(1,4-s)]$: x, I_7.

5.4.5 Rank Five

Derived Series $(n, n-2, 0)$

$\mathbf{ip}_{9,C}[\mathbf{L}=[[l]], l>5]$: x, $e/e_{1,6}$, where $e\in\{e_{1,3}, e_{2,5}\}$.

$\mathbf{ip}_{9,C}[s=1, m_0>4, l_{1,0}=1]]$: x, $e/e_{1,2}$, $e\in\{e_{1,4}, e_{3,5}\}$.

$\mathbf{ip}_{19,C}[s>3, \mathbf{j}=[0]]$: x, $(I_i)_{i=56}^{57}$.

Derived Series $(n, 1, 0)$

$\mathbf{ip}_{18,C}[s>3, \mathbf{j}=[1]]$: x, I_7, I_{59}.

5.4.6 Rank Four

Derived Series $(n, 0)$

$\mathbf{ip}_{18,C}[s > 4, \mathbf{j} = [0]]$: x, $(I_i)_{i=63}^{65}$.

5.5 Amaldis Groups Of Type D

The expressions J_i can be found in Subsection 4.2.4.

5.5.1 Lower Invariants

Derived Series (n)

$\mathbf{ip}_{2,D}[s = 1]$, $\mathbf{ip}_{5,D}[s = 1]$: z_y, J_4.

$\mathbf{ip}_{2,D}[s > 1]$, $\mathbf{ip}_{5,D}[s > 1]$: z_y.

$\mathbf{ip}_{17,D}$: z_x, z_y.

Derived Series $(n, n-1, n-2)$

$\mathbf{ip}_{5,D}[s = 0]$: z_x, z_y, J_2.

$\mathbf{ip}_{16,D}$: z_x, z_y, J_1.

Derived Series $(n, n-1, n-3, 3)$

$\mathbf{ip}_{1,D}[s > 1]$, $\mathbf{ip}_{3,D}[s > 2]$: z_y, J_2.

Derived Series $(n, n-2, n-4, 3)$

$\mathbf{ip}_{4,D}[s = 1]$: z_y, J_2, J_4.

$\mathbf{ip}_{4,D}[s > 1]$: z_y, J_2.

5.5.2 Rank Seven

Derived Series $(n, n-1, 3)$

$\mathbf{ip}_{20,D}[\sum s_i + l > 3]$: $J_{2,1}$.

Derived Series $(n, n-2, 3)$

$\mathbf{ip}_{21,D}[\sum s_i + s + l > 1]$: $J_{2,1}$.

5.5.3 Rank Six

Derived Series $(n, n-1, 3)$

$\mathbf{ip}_{19,D}[s > 1]$: x, $J_{2,1}$.

Derived Series $(n, 3)$

$\mathbf{ip}_{18,D}[s > 2]$: x, $J_{2,1}$.

5.6 Missing Definitions

The differential polynomial A in the invariant basis A^2B^{-3} of $\mathbf{ip}_{27}$ is defined as

$$\begin{aligned}
&z^6 z_{yy}^3 + 6z^5 z_{yy}^2(z_{xy} - z_y^2)+\\
&3z^4 z_{yy}(4z_y^4 + 4z_{xy}^2 + z_{xx}z_{yy} - 2z_y^2 z_{xy} - 8z_x z_y z_{yy})+\\
&2z^3(-4z_y^6 + 4z_{xy}^3 - 9z_x^2 z_{yy}^2 + 6z_y^2 z_{xy}^2 - 6z_y^4 z_{xy} + 30z_x z_y^3 z_{yy}\\
&\quad +3z_y^2 z_{xx} z_{yy} + 6z_{xx}z_{xy}z_{yy} - 30z_x z_y z_{yy} z_{xy})+\\
&3z^2(10z_y^4 z_{xx} + 10z_y^2 z_{xx} z_{xy} + z_{xx}^2 z_{yy} - 4z_x z_y z_{xx} z_{yy} - 32z_x z_y^3 z_{xy} - 8z_x z_y^5\\
&\quad -18z_x^2 z_{xy} z_{yy} + 4z_{xx} z_{xy}^2 + 46z_x^2 z_y^2 z_{yy} - 8z_x z_y z_{xy}^2)+\\
&z(12z_x z_y z_{xx} z_{xy} + 60z_x z_y^3 z_{xx} - 24z_x^2 z_y^4 - 156z_x^2 z_y^2 z_{xy} + 144z_x^3 z_y z_{yy}\\
&\quad -18z_x^2 z_{xx} z_{yy} - 36z_x^2 z_{xy}^2 + 6z_{xx}^2 z_{xy} + 12z_y^2 z_{xx}^2)+\\
&12z_x z_y z_{xx}^2 + 54z_x^4 z_{yy} - 18z_x^2 z_{xx} z_{xy} - 72z_x^3 z_y z_{xy} - 8z_x^3 z_y^3 + 30z_x^2 z_y^2 z_{xx} + z_{xx}^3,
\end{aligned}$$

and B is defined as

$$\begin{aligned}
&z^4 z_{yy}^2 + z^3(4z_{xy}z_{yy} - 4z_y^2 z_{yy})+\\
&z^2(4z_y^4 + 4z_{xy}^2 + 4z_y^2 z_{xy} + 2z_{xx}z_{yy} - 16z_x z_y z_{yy})+\\
&z(8z_x z_y^3 + 8z_y^2 z_{xx} - 12z_x^2 z_{yy} + 4z_{xx}z_{xy} - 8z_x z_y z_{xy})+\\
&z_{xx}^2 + 4z_x^2 z_y^2 + 8z_x z_y z_{xx} - 12z_x^2 z_{xy}.
\end{aligned}$$

Chapter 6

Examples and Conclusion

In this final chapter, we first show how to find the symmetry groups of given PDEs in our list of space groups. We demonstrate this on two classical PDEs, the Burgers equation and the Korteweg de Vries equation, in Section 6.1 ("Some Classical Equations"). In Section 6.2 ("Higher Invariants: Invariant Differentiation") we indicate the extension of Lie's higher invariants formula 4.1 to PDEs. Finally, in Section 6.3 ("Conclusion"), we explain what is left and how to proceed to complete group classification for second order PDEs in one dependent and two independent variables.

6.1 Some Classical Equations

In the following subsections we identify two classical PDEs with one dependent and two independent variables within the list of space groups: Burgers equation and the Korteweg de Vries equation.

The main difficulty originates from the fact that in general they will not be in canonical form, i.e. a suitable variable transformation has to be determined from the *actual variables* in which the equation is given, to the canonical variables corresponding to its symmetry type. The details of this procedure are described in Subsection 2.4 ("Basic Notions for Lie Algebras").

6.1.1 Symmetries of Burgers Equation

In order to find out where the symmetry group of the so called Burgers equation

$$z_x + zz_y + z_{yy} = 0$$

is contained in our space group list, we start with the following set of generators that was directly computed from the equation:

$$\{\partial_x, \partial_y, z\partial_z - 2x\partial_x - y\partial_y, x\partial_y + \partial_z, (xz - y)\partial_z - x^2\partial_x - xy\partial_y\}. \qquad \text{(BE)}$$

It is a five parameter group, so we will try to find it in the list given in Subsection 4.3.3 ("Groups with Five Generators") which contains all five parameter groups, except Amaldis groups of type B. In order to find a group similar to (BE), we consider its derived series, which is invariant under point transformations. For (BE) the dimension of its derived algebra is five, as can be seen by the table of commutators:

$$\begin{bmatrix} 0 & 0 & -2X_1 & X_2 & X_3 \\ 0 & 0 & -X_2 & 0 & -X_4 \\ 2X_1 & X_2 & 0 & -X_4 & -2X_5 \\ -X_2 & 0 & X_4 & 0 & 0 \\ -X_3 & X_4 & 2X_5 & 0 & 0 \end{bmatrix}.$$

Only the following four groups out of the list given in Subsection 4.3.3 have a derived algebra of dimension five:

$\mathbf{ip}_2[l = 1] = \{x\partial_y, x\partial_x - y\partial_y, y\partial_x, \partial_x, \partial_y\}$:

$$\begin{bmatrix} 0 & -2X_1 & X_2 & -X_5 & 0 \\ 2X_1 & 0 & -2X_3 & -X_4 & X_5 \\ -X_2 & 2X_3 & 0 & 0 & -X_4 \\ X_5 & X_4 & 0 & 0 & 0 \\ 0 & -X_5 & X_4 & 0 & 0 \end{bmatrix},$$

$\mathbf{ip}_{13} = \{\partial_x, \partial_y, x\partial_y + \partial_z, x\partial_x - y\partial_y - 2z\partial_z, y\partial_x - z^2\partial_z\}$:

$$\begin{bmatrix} 0 & 0 & X_2 & X_1 & 0 \\ 0 & 0 & 0 & -X_2 & X_1 \\ -X_2 & 0 & 0 & -2X_3 & X_4 \\ -X_1 & X_2 & 2X_3 & 0 & -2X_5 \\ 0 & -X_1 & -X_4 & 2X_5 & 0 \end{bmatrix},$$

$\mathbf{ip}^1_{2,A}[s=1] =$
$\{\partial_y, x\partial_y + \partial_z, \partial_x, 2x\partial_x + y\partial_y - z\partial_z, x^2\partial_x + xy\partial_y + (-xz+y)\partial_z\}$:

$$\begin{bmatrix} 0 & 0 & 0 & X_1 & X_2 \\ 0 & 0 & -X_1 & -X_2 & 0 \\ 0 & X_1 & 0 & 2X_3 & X_4 \\ -X_1 & X_2 & -2X_3 & 0 & 2X_5 \\ -X_2 & 0 & -X_4 & -2X_5 & 0 \end{bmatrix},$$

$\mathbf{ip}^2_{2,A}[s=1] = \{\partial_y, x\partial_y, \partial_x, 2x\partial_x + y\partial_y + z\partial_z, x^2\partial_x + xy\partial_y + z(x+cz^2)\partial_z\}$:

$$\begin{bmatrix} 0 & 0 & 0 & X_1 & X_2 \\ 0 & 0 & -X_1 & -X_2 & 0 \\ 0 & X_1 & 0 & 2X_3 & X_4 \\ -X_1 & X_2 & -2X_3 & 0 & 2X_5 \\ -X_2 & 0 & -X_4 & -2X_5 & 0 \end{bmatrix}.$$

Actually, we see that the third of these groups, $\mathbf{ip}^1_{2,A}[s=1]$, is generated by the same set of generators, up to sign, as (BE).

6.1.2 Symmetries of the Korteweg de Vries Equation

The Korteweg de Vries equation

$$z_x = zz_y + z_{yyy}$$

was considered the first time in the context of soliton solutions. Its symmetry group is generated by

$$\{\partial_x, \partial_y, 3\partial_y - \partial_z, 3x\partial_x + y\partial_y - 2z\partial_z\}. \qquad \text{(KdV)}$$

The dimension of its derived algebra is three, as can be seen by its table of commutators:

$$\begin{bmatrix} 0 & 0 & X_2 & 3X_1 \\ 0 & 0 & 0 & X_2 \\ -X_2 & 0 & 0 & -2X_3 \\ -3X_1 & -X_2 & 2X_3 & 0 \end{bmatrix}.$$

The twice derived algebra is of dimension one, which can be seen by the commutator table of $\{X_1, X_2, X_3\}$:

$$\begin{bmatrix} 0 & 0 & X_2 \\ 0 & 0 & 0 \\ -X_2 & 0 & 0 \end{bmatrix}.$$

Considering those the four parameter groups of Subsection 4.3.4 ("Groups with Four Generators") with the same derived series $(4, 3, 1, 0)$, we find that $\mathbf{ip}^1_{1,A}[s = 1, t = 1] = \{\partial_y, x\partial_y + \partial_z, \partial_x, x\partial_x + cy\partial_y + (c-1)z\partial_z\}$ is similar to (KdV). First of all we reorder $\mathbf{ip}^1_{1,A}[s = 1, t = 1]$ to

$$\{\partial_x, \partial_y, x\partial_y + \partial_z, x\partial_x + cy\partial_y + (c-1)z\partial_z\},$$

then we multiply the last generator by 3 and choose $c = \frac{1}{3}$, leading to

$$\{\partial_x, \partial_y, 3\partial_y + \partial_z, 3x\partial_x + y\partial_y - 2z\partial_z\}, \tag{IP1A1'}$$

which has the same commutator table as (KdV) and hence is isomorphic to it. In order to show that (IP1A1') is also similar to (KdV), we have to find a coordinate change $\overline{x} = x(x, y, z)$, $\overline{y} = y(x, y, z)$, $\overline{z} = z(x, y, z)$ such that the generators $\bar{X}_i = \bar{\xi}_{i1}\partial_x + \bar{\xi}_{i2}\partial_y + \bar{\xi}_{i3}\partial_z$ and $X_i = \xi_{i1}\partial_x + \xi_{i2}\partial_y + \xi_{i3}\partial_z$ of (KdV) and (IP1A1'), respectively, are related by

$$\begin{bmatrix} \bar{\xi}_{i1} \\ \bar{\xi}_{i2} \\ \bar{\xi}_{i3} \end{bmatrix} = \begin{bmatrix} \overline{x}_x & \overline{x}_y & \overline{x}_z \\ \overline{y}_x & \overline{y}_y & \overline{y}_z \\ \overline{z}_x & \overline{z}_y & \overline{z}_z \end{bmatrix} \cdot \begin{bmatrix} \xi_{i1} \\ \xi_{i2} \\ \xi_{i3} \end{bmatrix}, \quad i = 1, \ldots, 4.$$

The coordinate change $\overline{x} = x$, $\overline{y} = y$, $\overline{z} = -z$ achieves this, since

$$\begin{bmatrix} \bar{\xi}_{i1} \\ \bar{\xi}_{i2} \\ \bar{\xi}_{i3} \end{bmatrix} = \begin{bmatrix} \xi_{i1} \\ \xi_{i2} \\ -\xi_{i3} \end{bmatrix}, \quad i = 1, \ldots, 4.$$

leads to

$$\{\partial_{\overline{x}}, \partial_{\overline{y}}, 3\partial_{\overline{y}} - \partial_{\overline{z}}, 3x\partial_{\overline{x}} + y\partial_{\overline{y}} + 2z\partial_{\overline{z}}\},$$

i.e. to

$$\{\partial_{\overline{x}}, \partial_{\overline{y}}, 3\partial_{\overline{y}} - \partial_{\overline{z}}, 3\overline{x}\partial_{\overline{x}} + \overline{y}\partial_{\overline{y}} - 2\overline{z}\partial_{\overline{z}}\},$$

which is just (KdV) in the new coordinates $\overline{x}, \overline{y}, \overline{z}$.

Suppose we were given a group similar to (KdV), e. g.

$$\{\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4\}, \tag{KdV'}$$

where $\overline{X}_1 = -\overline{y}\partial_{\overline{y}} + \partial_{\overline{z}}$, $\overline{X}_2 = \overline{x}\partial_{\overline{x}}$, $\overline{X}_3 = \log(\frac{1}{\overline{y}})\overline{x}\partial_{\overline{x}} - \partial_{\overline{z}}$, $\overline{X}_4 = \overline{x}\log(\overline{x})\partial_{\overline{x}} - 3\overline{y}\log(\frac{1}{\overline{y}})\partial_{\overline{y}} + [5\log(\frac{1}{\overline{y}}) - 2\overline{z}]\partial_{\overline{z}}$, and would like to proof its similarity. First we note that (KdV') has the same commutator table as (KdV), hence it satisfies a necessary condition for similarity. Let us denote the generators of (KdV) by $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = 3\partial_y - \partial_z$, $X_4 = 3x\partial_x + y\partial_y - 2z\partial_z$. We note that X_1, X_2 and X_3 are unconnected, whereas

$$X_4 = 3xX_1 + (y - 2zx)X_2 + 2zX_3. \tag{Rel KdV}$$

We also note that the generators $\overline{X}_1$, $\overline{X}_2$ and $\overline{X}_3$ of (KdV') are unconnected, whereas

$$\overline{X}_4 = 3\log(\frac{1}{\overline{y}})\overline{X}_1 + [\log(\overline{x}) - 2\overline{z}\log(\frac{1}{\overline{y}}) + 2\log(\frac{1}{\overline{y}})^2]\overline{X}_2 + [2\overline{z} - 2\log(\frac{1}{\overline{y}})]\overline{X}_3. \tag{Rel KdV'}$$

Now, (Rel KdV) and (Rel KdV') lead to the following equations

$$\begin{aligned} 3x &= 3\log(\frac{1}{\overline{y}}), \\ y - 2xz &= \log(\overline{x}) - 2\overline{z}\log(\frac{1}{\overline{y}}) + 2\log(\frac{1}{\overline{y}})^2, \\ 2z &= 2\overline{z} - 2\log(\frac{1}{\overline{y}}). \end{aligned}$$

Solving them with respect to $\overline{x}$, $\overline{y}$ and $\overline{z}$ gives

$$\overline{x} = e^y,\ \overline{y} = e^{-x},\ \overline{z} = z + x,$$

a change of variables that establishes the similarity of (KdV) and (KdV').

6.1.3 The Heat Equation

The heat equation

$$z_{xx} - z_y = 0 \tag{Heat}$$

is an example of a second order PDE whose symmetry group is not contained in our list of space groups. Instead, we use it to demonstrate the computation of symmetry generators and their use in finding invariant solutions. A necessary and sufficient condition for an infinitesimal generator

$$X = \xi_1(x,y,z)\partial_x + \xi_2(x,y,z)\partial_y + \eta(x,y,z)\partial_z \tag{SG}$$

to be admitted by (Heat) is

$$X^{(2)}(z_{xx} - z_y) = 0 \text{ when (Heat).} \tag{D}$$

Hereby $X^{(2)}$ is the second order prolongation of X

$$X^{(2)} = \xi_1\partial_x + \xi_2\partial_y + \eta\partial_z + \eta_1^{(1)}\partial_{z_x} + \eta_2^{(1)}\partial_{z_y} + \eta_{11}^{(2)}\partial_{z_{xx}} + \eta_{12}^{(2)}\partial_{z_{xy}} + \eta_{22}^{(2)}\partial_{z_{yy}},$$

where $\eta_1^{(1)}, \eta_2^{(1)}, \eta_{11}^{(2)}, \eta_{12}^{(2)}, \eta_{22}^{(2)}$ are defined at the end of Subsection 2.2.1 ("Extended Infinitesimal Transformations"). Since the determining equation (D) is polynomial in $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$, and since ξ_1, ξ_2, η only depend on x, y, z, we may equate the coefficients of $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ (and their powers) in (D) to zero. The result is an overdetermined system of linear homogeneous equations in ξ_1, ξ_2, η and their partial derivatives up to order two, called *determining system*.

In our case, the determining equation (D) is

$$\eta_{11}^{(2)} - \eta_2^{(1)} = 0 \text{ when } z_{xx} = z_y. \tag{DE}$$

In the sequel we treat (DE) as $\eta_{11}^{(2)} - \eta_2^{(1)} = 0$, where every occurrence of z_{xx} is replaced by z_y. The coefficient of z_{xy} in (DE) is

$$-2(\xi_2)_x - 2z_x(\xi_2)_z.$$

Hence we know $(\xi_2)_x = (\xi_2)_z = 0$, i.e.

$$\xi_2 = \xi_2(y). \tag{K1}$$

Noting this, the coefficient of z_y in (DE) is

$$-2z_x(\xi_1)_z + (\xi_2)_y - 2(\xi_1)_x.$$

From this, we conclude

$$\begin{aligned} \xi_1 &= \xi_1(x,y), \\ (\xi_2)_y &= 2(\xi_1)_x. \end{aligned} \tag{K2}$$

By (K1), the second equation is equivalent to

$$\xi_1 = \{\frac{1}{2}(\xi_2)_y\}x + A(y). \tag{K3}$$

Considering (K1, K2, K3), we can view (DE) now as polynomial in z_x of degree two:

$$\eta_{zz}z_x^2 + \{\frac{1}{2}(\xi_2)_{yy} + A_y + 2\eta_{xz}\}z_x + \{\eta_{xx} - \eta_y\}. \tag{DE'}$$

Considering the coefficient of z_x^2, we have $\eta_{zz} = 0$, i.e.

$$\eta = f(x,y)z + g(x,y). \tag{K4}$$

Considering the coefficient of z_x^0 in (DE'), we have $\eta_{xx} = \eta_y$, i.e.

$$f_{xx} = f_y,\ g_{xx} = g_y. \tag{K5}$$

Finally, by considering the coefficient of z_x^1 in (DE'), we have

$$\frac{1}{2}(\xi_2)_{yy}x + A_y + 2f_x = 0. \tag{K6}$$

Summarizing, by (K1, K2, K3, K4, K5, K6) we now deal with the following system:

$$\xi_1 = \xi_1(x,y),\ \xi_2 = \xi_2(y),\ \eta = f(x,y)z + g(x,y), \tag{E1}$$

$$\xi_1 = \{\frac{1}{2}(\xi_2)_y\}x + A(y), \tag{E2}$$

$$f_{xx} = f_y,\ g_{xx} = g_y, \tag{E3}$$

$$\frac{1}{2}(\xi_2)_{yy}x + A_y + 2f_x = 0. \tag{E4}$$

We note that g, a solution of the original homogeneous equation (Heat), corresponds to the trivial infinite-parameter Lie group

$$\bar{x} = x,\ \bar{y} = y,\ \bar{z} = z + \varepsilon g(x,y),\ \text{where } g_{xx} = g_y,$$

that is admitted by every linear PDE. Nontrivial symmetries arise from the remaining equations. From (E4) we conclude that $f_{xxx} = 0$, i.e.

$$f(x,y) = C_1(y)x^2 + C_2(y)x + C_3(y).$$

By this and (E3) we conclude

$$2C_1(y) = C_1'(y)x^2 + C_2'(y)x + C_3'(y),$$

and hence we have

$$f(x,y) = c_1x^2 + c_2x + 2c_1y + c_3. \tag{E5}$$

By this and (E4) we have

$$\begin{aligned}\frac{1}{2}(\xi_2)_{yy} + 4c_1 &= 0,\\ A_y + 2c_2 &= 0,\end{aligned}$$

and hence

$$\xi_2(y) = -4c_1y^2 + b_1y + b_2, \tag{E6}$$

$$A(y) = -2c_2y + \bar{a}. \tag{E7}$$

Summarizing, by (E1, E2, E5, E6, E7) the final result is

$$\begin{aligned}\xi_1(x,y) &= -4c_1xy + \frac{1}{2}b_1x - 2c_2y + a,\\ \xi_2(y) &= -4c_1y^2 + b_1y + b_2,\\ \eta(x,y,z) &= \{c_1x^2 + c_2x + 2c_1y + c_3\}z.\end{aligned}$$

The infinitesimal generator

$$X = \xi_1(x,y)\partial_x + \xi_2(y)\partial_y + \eta(x,y,z)\partial_z$$

hence represents a six-parameter group in the parameters $a, b_1, b_2, c_1, c_2, c_3$. The same Lie algebra is spanned by the following six generators, each of which corresponds to a one-parameter group:

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + 2y\partial_y,\ X_4 = 4xy\partial_x + 4y^2\partial_y - (x^2+2y)z\partial_z,$$
$$X_5 = 2y\partial_x - xz\partial_z,\ X_6 = z\partial_z.$$

The commutator table corresponding to $L = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ is

$$\begin{bmatrix} 0 & 0 & X_1 & 2X_5 & -X_6 & 0\\ 0 & 0 & 2X_2 & 4X_3 - 2X_6 & 2X_1 & 0\\ -X_1 & -2X_2 & 0 & 2X_4 & X_5 & 0\\ -2X_5 & -4X_3 + 2X_6 & -2X_4 & 0 & 0 & 0\\ X_6 & -2X_1 & -X_5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

its derived series is (6).

Let us consider the infinitesimal generator X_4, which corresponds to the parameter c_1. The one-parameter Lie group of transformations

$$\bar{x}(x,y,z,\epsilon),\ \bar{y}(x,y,z,\epsilon),\ \bar{z}(x,y,z,\epsilon) \tag{LT}$$

corresponding to $X_4 = 4xy\partial_x + 4y^2\partial_y - (x^2+2y)z\partial_z$ is obtained by solving the initial value problem

$$(\bar{x},\bar{y},\bar{z})[\epsilon=0] = (x,y,z) \tag{IVP}$$

for the following first order system of ODEs:

$$\frac{d\bar{x}}{d\varepsilon} = 4\bar{x}\bar{y}, \tag{ODEx}$$

$$\frac{d\bar{y}}{d\varepsilon} = 4\bar{y}^2, \tag{ODEy}$$

$$\frac{d\bar{z}}{d\varepsilon} = -(\bar{x}^2+2\bar{y})\bar{z}. \tag{ODEz}$$

The solution of (ODEy) is $\bar{y} = \frac{1}{C-4\epsilon}$, and by (IVP) we obtain

$$\bar{y}(x,y,z,\epsilon) = \frac{y}{1-4\epsilon y}. \tag{SOLy}$$

By this and (ODEx) we get $\bar{x} = \frac{C}{1-4\epsilon y}$, and by (IVP) we obtain

$$\bar{x}(x,y,z,\epsilon) = \frac{x}{1-4\epsilon y}. \tag{SOLx}$$

Similarly, by (SOLx, SOLy, ODEz) and (IVP) we obtain

$$\bar{z}(x,y,z,\epsilon) = z\sqrt{1-4\epsilon y}\exp(-\frac{\epsilon x^2}{1-4\epsilon y}). \tag{SOLz}$$

Every invariant solution $z = \Phi(x,y)$ of (Heat) corresponding to X_4 satisfies

$$X_4(z-\Phi(x,y)) = 0 \text{ when } z-\Phi(x,y),$$

i.e.

$$4xy\frac{\partial\Phi}{\partial x} + 4y^2\frac{\partial\Phi}{\partial y} = -(x^2+2y)\Phi. \tag{IC}$$

We solve (IC) by solving the corresponding characteristic equation

$$\frac{dx}{4xy} = \frac{dy}{4y^2} = \frac{dz}{-(x^2+2y)z}$$

which has the two invariants

$$\frac{x}{y}, \text{ and } z\sqrt{y}e^{x^2/4y}.$$

The solution of (Heat) is now defined by the invariant form

$$z\sqrt{y}e^{x^2/4y} = \phi(\frac{x}{y}),$$

or, in explicit form,

$$z = \Phi(x,y) = \frac{1}{\sqrt{y}}e^{-x^2/4y}\phi(\zeta), \tag{ES}$$

where $\zeta = \frac{x}{y}$ is the similarity variable. Substitution of (ES) into (Heat) leads to $\phi''(\zeta) = 0$. Hence, invariant solutions of (Heat) resulting from X_4 are of the form

$$z = \Phi(x,y) = \frac{1}{\sqrt{y}}e^{-x^2/4y}\{C_1 + C_2\frac{x}{y}\}.$$

For any solution $z = \Phi(x,y)$ of (Heat), that is not invariant under X_4, we find a one-parameter family of solutions $z = \phi(x,y,\epsilon)$ generated by X_4: Let

$$\begin{aligned} x^* &= \bar{x}(x,y,z,\epsilon) = \frac{x}{1-4\epsilon y}, \\ y^* &= \bar{y}(x,y,z,\epsilon) = \frac{y}{1-4\epsilon y}, \\ z^* &= \Phi(\bar{x},\bar{y}). \end{aligned}$$

By $\bar{z}(\cdot,\cdot,\cdot,-\epsilon)$ we denote the third component of the inverse transformation corresponding to X_4. Then $z = \phi(x,y,\epsilon) = \bar{z}(x^*,y^*,z^*,-\epsilon) =$

$$\Phi(\frac{x}{1-4\epsilon y}, \frac{y}{1-4\epsilon y})\frac{1}{\sqrt{1-4\epsilon y}}\exp(\frac{\epsilon x^2}{1-4\epsilon y}).$$

6.2 Higher Invariants: Invariant Differentiation

In this section, we use the convention that we sum over multiply occurring indices in an expression. For ordinary differential equations, Lie showed that all invariants of order $2, 3, \ldots$ can be found merely by differentiation, provided that the invariants u, v of order 0 and 1, respectively, are known.

Theorem (Invariant Differentiation): Let G be a one-parameter point transformation group in the plane with the generator X. Let $u(x, y)$ and $v(x, y, y')$ be an invariant and a first order differential invariant of G. Then

$$w = \frac{dv}{du} = \frac{v_x + y'v_y + y''v_{y'}}{u_x + y'u_y} = \frac{D_x(v)}{D_x(u)} \tag{6.1}$$

is a second-order differential invariant of the group G.

Since $X^{(2)}F(x, y, y', y'') = 0$ has precisely three independent solutions, all differential invariants F, $ord(F) \leq 2$, are given by $F = \Phi(u, v, w)$. We now rewrite (6.1) in the form $w = \mathcal{D}(v)$ with the operator $\mathcal{D}$ defined by

$$\mathcal{D} = \lambda D_x, \text{ where } \lambda = \frac{1}{D_x(u)}. \tag{6.2}$$

Definition: The operator $\mathcal{D}$ in (6.2), where $u(x, y)$ is any invariant of G, is called an *invariant differentiation* for the group G.

The significance of this definition is disclosed by the following statement.

Theorem (Invariant Basis): The invariant differentiation $\mathcal{D}$ converts any differential invariant F of the group G into a differential invariant $\mathcal{D}(F)$ of G. Furthermore, any differential invariant F, $ord(F) = s+1$, can be expressed as a function of $u(x, y)$, $v(x, y, y')$ and successive invariant derivatives of the first-order differential invariant v:

$$F = \Phi(u, v, \mathcal{D}(v), \mathcal{D}^2(v), \ldots, \mathcal{D}^s(v)).$$

Similar results hold also in the case of many variables and multi-parameter groups [13]. To formulate this generalization, we first note that the coefficient $\lambda = 1/D_x(u)$ of the invariant differentiation (6.2) satisfies the equation

$$X^{(1)}(\lambda) = \lambda D_x(\xi). \tag{6.3}$$

Indeed, one can verify by straightforward computation that the following operator identity holds for the infinite-times extended generator X:

$$XD_i - D_iX = -D_i(\xi^j)D_j, \tag{6.4}$$

where D_i is the operator of total differentiation w.r.t. x_i

$$D_i = \partial_{x_i} + u_i^\alpha \partial_{u^\alpha} + u_{i,i_1}^\alpha \partial_{u_{i_1}^\alpha} + u_{i,i_1,i_2}^\alpha \partial_{u_{i_1,i_2}^\alpha} + \dots, \quad i = 1, \dots, n.$$

We now consider the one-dimensional case and apply (6.4), written in the form $XD_x = D_xX - D_x(\xi)D_x$, to $\lambda(x, y, y') = 1/D_x(u)$. Invoking that $u(x, y)$ is an invariant, i.e. $X(u) = 0$, we arrive at (6.3) as follows:

$$X^{(1)}(\lambda) = -\frac{X(D_x(u))}{(D_x(u))^2} = -\frac{D_xX(u) - D_x(\xi)D_x(u)}{(D_x(u))^2} = \frac{D_x(\xi)}{D_x(u)} = \lambda D_x(\xi).$$

The generalization to the higher-dimensional case is given by the following invariant derivations. Let G_r be any r-parameter group of point transformations with infinitesimal generators

$$X_\upsilon = \xi_\upsilon^i(x, u)\frac{\partial}{\partial x^i} + \eta_\upsilon^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \quad \upsilon = 1, \dots, r,$$

where $x = x_1, \dots, x_n$, $u = u^1, \dots, u^m$. Then there exist n independent invariant derivations

$$\mathcal{D} = \lambda^i D_i,$$

where $\lambda^i(x, u, u^{(1)}, u^{(2)}, \dots)$ are differential functions determined from equations similar to (6.3):

$$X_\upsilon(\lambda_i) = \lambda^j D_j(\xi_\nu^i), \quad i = 1, \dots, n; \quad \upsilon = 1, \dots, r.$$

6.3 Conclusion

We repeat the goal of any group classification problem: it aims to obtain a complete survey of all possible symmetry groups for a class of given DEs, in our case second order PDEs in one dependent and two independent variables. The starting point of this approach was the listing of space groups given in Chapter 3 ("The Space Point Groups"), whose differential invariants determine the general form of a PDE that may be invariant under the respective group. The whole strategy to tackle the classification problem, as given in Chapter 1 ("Introduction"), was as follows:

1) List all finite continuous transformation groups of the three dimensional space in coordinates x, y, and z.

2) Find all differential invariants of the groups given in 1) where z depends on x and y.

3) Determine the group types, i.e. by using each invariant from 2), find criterions that allow to identify the symmetry group for a given DE from our class.

In Chapter 3 ("The Space Point Groups"), we solved point 1) of the above strategy by providing a listing of all point transformation groups of the three dimensional space which is based on work by S. Lie [7, 8] and U. Amaldi [1]. This listing is claimed to be complete, but not necessarily disjoint. In Chapter 4 ("Differential Invariants of Order Two") and Chapter 5 ("Lower Invariants"), we solved a large part of point 2) of the above strategy by presenting a list of computed differential and lower invariants of order two of many space groups listed in Chapter 3. The exceptions are the groups given in Subsection 4.3.8 ("Groups whose Invariants were not found") and the Amaldi groups of type B. Those are, due to the huge number of space groups, not handled within the frame of this thesis. Consequently the same holds for point 3) of the above strategy.

As a result, further work on the solution of this problem should focus on

1*) the calculation of all differential invariants of order two of Amaldis groups of type B,

2*) point 3) of the main strategy.

This last step would be accomplished by first computing the Janet base of the determining system of all differential invariants, then by applying a general point transformation to it. Since that in general destroys the Janet base property, one has to reestablish it by applying the algorithm Janet base again. Thereby, a classification of Janet bases for determining systems of DEs from the indicated class would be achieved. We also remark that the identification of multiple occurrences of group types in our space group list is the natural by-product of this last step.

Two other problems for which this work may be the starting point should be mentioned: the implicit description of the differential invariants in one dependent and two independent variables of order higher than two may be accomplished by providing the invariant differentiations indicated in Section 6.1 ("Higher Invariants: Invariant Differentiation"). Finally, the classification problem for systems of ODEs in two dependent and one independent variable $y(x)$, $z(x)$ might be accomplished by starting with the groups of the space listed in this work, using the prolongation formulas for two dependent and one independent variable, instead.

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